Open and closed sets

Implicitly, an inequality means both sides are taken to be real, unless otherwise specified or unless context makes a different meaning clear.

<u>Temporary Def.</u>: Let (E, d) be a metric space, $p_0 \in E$, $r > 0$.

The <u>openball</u> (one word) of radius r, centered at p_0, is

$$B_r(p_0) = \{p \in E \mid d(p, p_0) < r\}.$$

The <u>closedball</u> (one word) of radius r, centered at p_0, is

$$\bar{B}_r(p) = \{p \in E \mid d(p, p_0) \leq r\}$$

Note that the bar goes <u>only</u> over the letter B in this notation.

This part is not temporary

A <u>ball</u> ~~means~~ is an openball or closedball of some center and radius.

<u>Note</u>: In mathematics, the words "ball" and "sphere" are not interchangeable! The <u>sphere</u> of radius r, centered at p_0, is $\{p \in E \mid d(p, p_0) = r\}$.

<u>Examples</u>:

① In $\mathbb{R}$,

$$B_r(x_0) = (x_0 - r, x_0 + r) = \{x \in \mathbb{R} \mid x_0 - r < x < x_0 + r\}$$

$$\bar{B}_r(x_0) = [x_0 - r, x_0 + r] = \{x \in \mathbb{R} \mid x_0 - r \leq x \leq x_0 + r\}$$

② In $(\mathbb{R}^2, d_{Euc})$ (where d_{Euc} is the Euclidean, or ℓ^2, metric),

$B_r((x_0, y_0))$ = open disk centered at (x_0, y_0)

$\bar{B}_r((x_0, y_0))$ = clsd disk = (open disk) $\cup$ (boundary circle)

③ In $(\mathbb{R}^3, d_{Euc})$: Similar to $(\mathbb{R}^2, d_{Euc})$, ~~but~~ with "disk" replaced by "solid ball" (informal terminology).

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<u>Note</u>: In any normed vector space, $B_r(p_0)$ is a translate of $B_r(0)$:

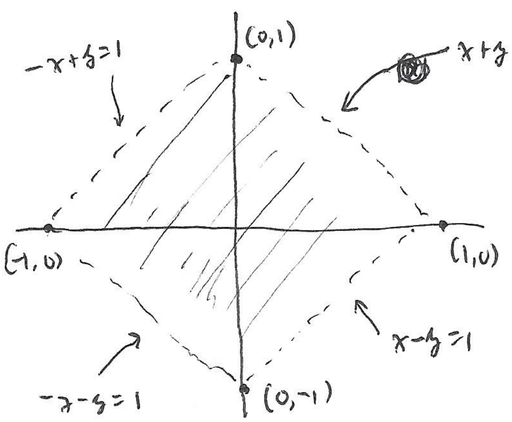
$$B_r(p_0) = \{p_0 + x \mid x \in B_r(0)\}.$$

Analogous statement holds for $\bar{B}_r(p_0)$. So, to understand all balls in a normed v.s., it suffices to understand balls centered at 0.

Examples, cont'd

(4) ($\mathbb{R}^2$, ℓ^1 metric).

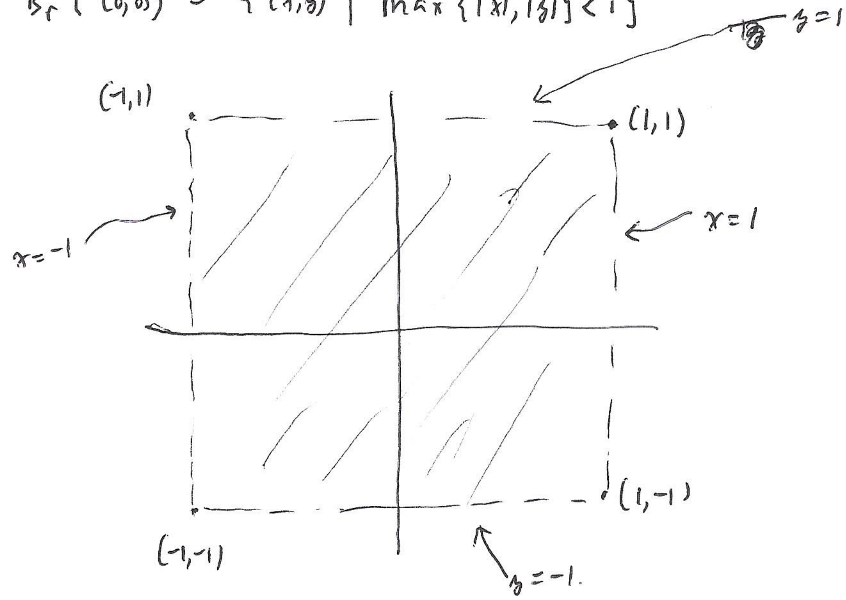
$$B_r((0,0)) = \{(x,y) \mid |x| + |y| < r\}$$



Picture for $r=1$. Openballs of different radius centered at $(0,0)$ are just scaled-up (if $r>1$) or scaled-down (if $r<1$) versions of this. For closedballs, the dotted lines would be included.

(5) $(\mathbb{R}^2, \ell^\infty \text{ metric})$.

$B_r((0,0)) = \{(x,y) \mid \max\{|x|,|y|\} < 1\}$



Picture of $B_1((0,0))$.

The same comments as in Ex. 4 apply for different radii and for closed balls.

Def: Let (E,d) be a metric space. A subset $U \subset E$ is *open* if, $\forall p \in E$, $\exists r>0$ s.t. $B_r(p) \subset E$.

A set $V \subset E$ is *closed* if its complement $\complement V$ is open.

Note: "Closed" is not the opposite (or negation) of open. "Not open" does not mean "closed"! Both open sets and closed sets are *very special* types of sets. In most metric spaces, most sets are neither open nor closed.

Remark: Given (E,d), let

$$\mathcal{T}_{op} = \{\text{open subsets of } E\} \subset P(E)$$

$$\mathcal{T}_{cl} = \{\text{closed subsets of } E\} \subset P(E)$$

(temporary notation). Complementation gives a 1-1 correspondence

$$\begin{array}{ccc} \mathcal{T}_{op} & \longleftrightarrow & \mathcal{T}_{cl} \\ U & \longmapsto & \complement U \\ \complement V & \longmapsfrom & V \end{array}$$

Thus, for any statement about open sets, there will be a corresponding statement about closed sets (not generally the same statement with "open" replaced by "closed"; just *some* statement that can be worked out by tracing-through definitions.

Here are two "sister" propositions that we will prove simultaneously:

(13.3) 2015 | (15.4) 2016 | (13.5) 2017

Prop. 1: Let (E, d) be a metric space. Then

(i) $\emptyset$ is open

(ii) E is open

(iii) Arb. union of open sets is open. I.e. if $\{U_\alpha\}_{\alpha \in A}$ is a collection of open subsets of E indexed by a set A, then $\bigcup_{\alpha \in A} U_\alpha$ is open.

(iv) Any finite intersection of open sets is open. (I.e., given open sets $U_1, \dots, U_n$, $\bigcap_{i=1}^{n} U_i$ is open.)

(v) Any open ball is open.

Rmk: If $\mathcal{T} \subset \mathcal{P}(X)$ (any set) is any collection of subsets s.t. (i) $\emptyset \in \mathcal{T}$, (ii) $E \in \mathcal{T}$, (iii) Arb. union of elts of $\mathcal{T}$ is in $\mathcal{T}$, and (iv) any finite intersection of elts of $\mathcal{T}$ is in $\mathcal{T}$, we call $\mathcal{T}$ a topology on X.

Prop. 2 Let (E, d) be a metric space. Then

(i)' $\emptyset$ is clsd

(ii)' E is clsd

(iii)' Arb. intersection of clsd sets is clsd.

(iv)' Finite union of clsd sets is clsd.

(v)' Any closed ball is clsd.

Pf of Props 1 and 2:

(a) $\emptyset$ is open, vacuously. Hence $E = \complement\emptyset$ is closed. This proves (i) & (ii)'.

(b) E is open, tautologically. Hence $\emptyset = \complement E$ is clsd. This proves (ii) & (i)'.

(13.4) 2015 | (15.5) 2016 ~~16.1~~ | (13.6) 2017

(c) Let $\{U_\alpha\}_{\alpha \in A}$ be ~~a collection~~ a collection of open sets.

Let $U = \bigcup_{\alpha \in A} U_\alpha$. If $U = \emptyset$, then U open, by (i).

So suppose $U \neq \emptyset$, and let $p \in U$. Then $p \in U_{\alpha_0}$ for some α_0; select such α_0. Since U_{α_0} is open $\exists\, r > 0$ s.t.

$$\begin{aligned} B_r(p) &\subset U_{\alpha_0} \\ &\subset \bigcup_{\alpha \in A} U_\alpha = U. \end{aligned}$$

Hence $B_r(p) \subset U$. Hence U is open. This proves (iii)

Now let $\{V_\alpha\}_{\alpha \in A}$ be a collection of closed sets, and let $V = \bigcap_\alpha V_\alpha$. Since V_α is clsd, $\complement V_\alpha$ is open, $\forall \alpha$. So $\bigcup_\alpha \complement V_\alpha$ open, by (iii). Hence $\complement\left(\bigcup_\alpha \complement V_\alpha\right)$ is clsd. But

$$\complement\left(\bigcup_\alpha \complement V_\alpha\right) = \bigcap_\alpha \complement(\complement V_\alpha) = \bigcap_\alpha V_\alpha = V,$$

so V is clsd. This proves (iii)'

(d) Let $U_1, \dots, U_n$ be open, let $U = \bigcap_{i=1}^n U_i$, and let $p \in U$. Then $p \in U_i$, $1 \leq i \leq n$. Since U_i open, $\exists\, r_i > 0$ s.t. $B_{r_i}(p) \subset U_i$ $\forall i$. Let $r = \min\{r_1, \dots, r_n\}$. Then $\forall i$, $r \leq r_i$, $\Rightarrow$ $B_r(p) \subset B_{r_i}(p) \subset U_i$. Hence $B_r(p) \subset \bigcap_i U_i = U$. $\therefore$ U is open. This proves (iv).

Now let $V_1, \dots, V_n$ be clsd, and let $V = \bigcup_{i=1}^n V_i$. Since V_i clsd, $\complement V_i$ open, so by (iv), $\bigcup_{i=1}^n \complement V_i$ open. Hence $\complement\left(\bigcup_{i=1}^n \complement V_i\right)$ clsd. But

$$\complement\left(\bigcup_i \complement V_i\right) = \bigcap_i \complement(\complement V_i) = \bigcap_i V_i = V.$$

Hence V clsd. This proves (iv)!