MAA 4211

CONTINUITY, IMAGES, AND INVERSE IMAGES

By the end of the semester we will have proven several theorems of the form: "Under a continuous function, the image (or inverse image) of a set with a certain property also has that property." (Some of these theorems are about images and some are about inverse images; none of the theorems is about both.) For example, under a continuous function, the inverse image of an open subset of the is always an open subset of the domain. We may think of these theorems as asserting that, when dealing with continuous functions, certain properties of sets are preserved in one direction or the other; i.e. either for "forward" images or inverse images.

This handout provides, for easy reference, the short list of which specific assertions of this type are true, and a longer list of the many that are false.

For the rest of this handout, (X, d_X) and (Y, d_Y) are metric spaces, $f : X \to Y$ is a **continuous** function, the letter U will be used only for subsets of X, and the letter V will be used only for subsets of Y. Thus $f(U)$ will always be a subset of Y, and $f^{-1}(V)$ will always be a subset of X.

Properties that are preserved in one direction or the other

First, here are some short-hand names to help you remember the true facts. Precise versions of the facts are given after the list.

1. "f^{-1}(open) = open"
2. "f^{-1}(closed) = closed"
3. "f(compact) = compact"
4. "f(connected) = connected"

What these short-hand names stand for are the following:

1. If $f : X \to Y$ is continuous and $V \subset Y$ is open, then $f^{-1}(V)$ is open.

   Another good wording: Under a continuous function, the inverse image of an open set is open.

2. If $f : X \to Y$ is continuous and $V \subset Y$ is closed, then $f^{-1}(V)$ is closed.

   Another good wording: Under a continuous function, the inverse image of a closed set is closed.

3. If $f : X \to Y$ is continuous and $U \subset Y$ is compact, then $f(U)$ is compact.

   Another good wording: A continuous function maps compact sets to compact sets.

   Less precise wording: "The continuous image of a compact set is compact."

   (This less-precise wording involves an abuse of terminology; an image is not an object that can be continuous. Nonetheless, I have found this mantra an easy way to hold the precise fact in my head.)

4. If $f : X \to Y$ is continuous and $U \subset Y$ is connected, then $f(U)$ is connected.

   Another good wording: A continuous function maps connected sets to connected sets.

   Less precise wording: "The continuous image of a connected set is connected."

   (Another abuse of terminology, but again a mantra I have found useful.)

Properties that are *not* preserved at least one direction

Looking at the list of preserved properties and the directions in which they are preserved, some natural questions are: Are any of these properties preserved in both directions? Are there similar true statements we can make, either for forward images or inverse images, about two other metric properties we have studied: boundedness and completeness?

The answer is **no** to both questions (if we assume nothing about f other than that it is continuous). For example, the image of an open set under a continuous function is not necessarily open. This will be abbreviated below as "$f(\text{open}) \neq \text{open}$". This short-hand stands for the logical negation of what "$f(\text{open}) = \text{open}$" would mean. I.e. "$f(\text{open}) \neq \text{open}$" means that for *some* metric space X, *some* metric space Y, *some* continuous function $f : X \to Y$, and *some* open set $U \subset X$, the set $f(U)$ is not open in Y. Said another way, it means that there exists *at least one* counterexample to the assertion "$f(\text{open}) = \text{open}$". (It happens that counterexamples abound, but that is not what the statement "$f(\text{open}) \neq \text{open}$" is saying.)

Similar short-hand will be used for other statements in the list below, so **be sure you understand what this short-hand means**. In particular, the statement "$f(\text{open}) \neq \text{open}$" does *not* mean that, under a continuous function, the image of an open set is *never* open. It does *not* mean that for every continuous function $f : X \to Y$ there exists an open set $U \subset X$ for which $f(U)$ is not open. It does *not* mean that for every pair of metric spaces X and Y, there is a continuous function f and an open set $U \subset X$ such that $f(U)$ is not open. It does *not* even mean that for every metric space X (or Y) there exists a metric space Y (or X), and a continuous function $f : X \to Y$, and an open set $U \subset X$, such that $f(U)$ is not open. It means only that for *some* X, Y, *some* continuous $f : X \to Y$, and *some* open $U \subset X$, the set $f(U)$ is not open in Y.

With this short-hand understood, here is a listing of the *non*-preservation of various set-properties when we take forward and/or inverse images under a continuous function.

1. "f(open) $\neq$ open"
2. "f(closed) $\neq$ closed"
3. "f^{-1}(compact) $\neq$ compact"
4. "f^{-1}(connected) $\neq$ connected"
5. "f(bounded) $\neq$ bounded"
6. "f^{-1}(bounded) $\neq$ bounded"
7. "f(complete) $\neq$ complete"
8. "f^{-1}(complete) $\neq$ complete"

What goes wrong: some counterexamples illustrating the facts above

In all the examples below, the metric on $\mathbf{R}$ is assumed to be the standard one unless otherwise specified.

1. Let $X = Y = \mathbf{R}$ and define f by $f(x) = x^2$. Let $V = [1, 4]$, or any other interval contained in $(0, \infty)$. Then V is connected, but $f^{-1}(V)$ is not. (E.g. $f^{-1}([1, 4]) = [-2, -1] \bigcup [1, 2]$.)
2. Let $X = Y = \mathbf{R}$ and define f by $f(x) = \frac{1}{1+x^2}$.
   (a) Let $U = \mathbf{R}$. Then $f(U) = (0, 1] \subset \mathbf{R}$. Thus U is open, closed, and complete, but $f(U)$ is none of these. (Note: "closed and complete" is redundant, since "complete" implies "closed". The examples in this handout include several such redundancies, in order to make immediate contact with the statements on the list above.)

   The next two examples illustrate that we can obtain counterexamples using proper subsets of X, not just with $U = X$.

   (b) Let $U = [0, \infty)$. Then again $f(U) = (0, 1]$. In this case U is a proper subset of X that is closed and complete, but $f(U)$ is neither closed nor complete.
   (c) Let $U = (-1, 1)$. Then $f(U) = (\frac{1}{2}, 1]$. Thus U is a proper subset of X that is open, but $f(U)$ is not open. (The same would have been true had we taken U to be any open interval containing 0.)
   (d) Let $V = (0, 1]$. Then V is bounded, but $f^{-1}(V) = \mathbf{R}$, which is unbounded.
   (e) Let $V = [0, 1]$. Then V is compact, but $f^{-1}(V) = \mathbf{R}$, which is not compact.

(f) Let $V = [0, \frac{1}{2}]$. Then V is compact and connected, but $f^{-1}(V)$ is neither.

3. Let $X = (0, 1]$ (as a subspace of $\mathbf{R}$), $Y = \mathbf{R}$, and define $g : X \to Y$ by $g(x) = \frac{1}{x}$.

   (a) Let $U = X$. Then U is bounded but $f(U) = [1, \infty)$, which is not bounded. (Exercise: Since $U = X$, U is also closed in X. Find the mistake in the following argument. "U is closed and bounded. Since $U \subset \mathbf{R}$, the Heine-Borel Theorem implies that U is compact. Therefore $f(U)$ is compact, hence bounded.")

   (b) Let $V = \mathbf{R}$. Then V is complete but $f^{-1}(V) = (0, 1]$ is not.

4. Let $X = (0, 1]$ (as a subspace of $\mathbf{R}$), $Y = \mathbf{R}$, and define $f : X \to Y$ by $f(x) = x$.

   (a) Let $U = X$. Then U is open and closed in X, but $f(U)$ is neither open nor closed in Y.

   (b) Let $V = \mathbf{R}$ or $V = [0, 1]$. Then V is complete but $f^{-1}(V) = X$, which is not complete.

   (c) Let $V = [0, \frac{1}{2}]$. Then V is compact but $f^{-1}(V) = (0, \frac{1}{2}]$, which is not compact.

5. Let X, Y be any nonempty metric spaces, $c \in Y$, f the constant map defined by $f(x) = c$ for all $x \in X$. Let $V = \{c\}$. Then V is compact, complete, and connected. But $f^{-1}(V) = X$, which is not compact if X is not compact, not complete if X is not complete, and not connected if X is not connected.

6. Let $X = \mathbf{R}$ but with the discrete metric d defined by $d(p, q) = \left\{ \begin{array}{ll} 1 & \text{if } p \neq q, \\ 0 & \text{if } p = q \end{array} \right\}$. Let $Y = \mathbf{R}$ (with the standard metric). Define $f : X \to Y$ by $f(x) = x$.

   (a) Let $U = \mathbf{Q}$ or $U = \mathbf{R} \setminus \mathbf{Q}$. Recall that every subset of (X, d) is both open and closed. Thus U is open and closed in (X, d), but $f(U) = U$ is neither open nor closed in Y.

   (b) Let $V = [0, 1]$. Then V is compact (as a subspace of Y), but $f^{-1}(V) = [0, 1]$—the same *set* as V, but now viewed as a subspace of (X, d)—is not compact. (The only sets of (X, d) that are compact are finite sets.)

7. Let $X = \mathbf{Q}$ (as a subspace of $\mathbf{R}$), $Y = \mathbf{R}$, and define $f : X \to Y$ by $f(x) = x$. Let $V = \mathbf{R}$ or any closed interval of nonzero length in $\mathbf{R}$. Then V is complete, but $f^{-1}(V) = V \bigcap \mathbf{Q}$ is not complete. If the interval V is taken to be bounded (as well as closed), then V is compact, but $f^{-1}(V)$ is not compact.

8. (A more exotic example.) Let $X = [-4, \infty)$, $Y = \mathbf{R}^2$. Define $f : X \to Y$ by

$$f(t) = \left\{ \begin{array}{ll} (0, -t-1), & -4 \leq t \leq 0, \\ \left(\frac{t^2-1}{t^2+1}, \frac{2t}{t^2+1}\right), & t > 0. \end{array} \right.$$

It is easily verified that f is continuous. The image $f(X)$ is the union of the line-segment $S = \{0\} \times [-1, 3]$ (lying on the y-axis) and the semicircle $C = \{(x, y) \in \mathbf{R}^2 \mid x^2 + y^2 = 1, x \geq 0\}$. (For $-2 \leq t \leq 0$, f traces out S from top to bottom. For $0 \leq t < \infty$, f traces out $C \setminus \{(0, 1)\}$. As $t \to \infty$, $f(t)$ gets arbitrarily close to $(0, 1)$, but never reaches $(0, 1)$. But $(0, 1)$ is still in the image of f; specifically $f(-2) = (0, 1)$.)

Let W be a small, closed disk of radius $\epsilon < 1$, centered at $(0, 1)$, and let $V = f(X) \bigcap W$. (V looks something like the symbol " $\vdash$ ", with but the horizontal part replaced by a small arc of C). Then V is connected and compact, but $f^{-1}(V)$ is neither connected nor compact (it is closed but not bounded). The set $f^{-1}(V)$ has two connected components, one of which is the compact interval $[-2 - \epsilon, -2 + \epsilon]$, and the other of which is an interval of the form $[a, \infty)$, $a \in \mathbf{R}_+$. (If we vary ϵ, then $a \to \infty$ as $\epsilon \to 0$.) Taking the inverse image "tears" the part of V that lies on C from the vertical part of V, like yanking an arm from its socket (except that we also make the arm infinitely long after the yanking).

In the examples 1 and 2(f), in which V was connected, the failure of $f^{-1}(V)$ to be a connected was a result of the non-injectivity of those f's. But the f in the present example *is* one-to-one. So the statement "f^{-1}(connected) = connected" would be false even if the only f's we considered were injective as well as continuous.

A variant of this example: replace $[-4, \infty)$ by $[-4, \pi)$ and, for $0 < t < \pi$, define $f(t) = (\cos(t - \pi/2), \sin(t - \pi/2)) = (\sin t, -\cos t)$. (Keep f the same as before on $[-4, 0]$.) The image of this f is the same as the old one's. For the same V as before, $f^{-1}(V)$ is again neither connected nor compact. This inverse image again has two connected components, one of which is the compact interval $[-2 - \epsilon, -2 + \epsilon]$, and the other of which is an interval $[a, \pi)$ for some $a \in (0, \pi)$. The latter component is closed in X and bounded, yet not compact. (Make sure you understand why this paradox does not contradict the Heine-Borel Theorem; see the Exercise in example 3(a).) For this new f, our cruel yanking of the arm from its socket at least does not stretch the arm to infinite length.