MAA 4211, Fall 2018—Assignment 1's non-book problems

B1. **"Empty" functions**. Let X and Y be sets. In this problem we consider functions from X to Y when X or Y is empty. For this, it is convenient to use the set-theoretic definition of "function", which amounts to defining a function to *be* its graph. In this definition, a function f from X to Y is a subset of $X \times Y$ with the property that for each $x \in X$, there exists a unique $y \in Y$ such that $(x, y) \in f$.

(a) Show that there exists a unique function $f : \emptyset \to Y$.

(b) Show that if $X \neq \emptyset$, there does not exist any function $f : X \to \emptyset$.

The function in part (a) is called an *empty function*, or more specifically "the empty Y-valued function". (The answer to the question "What subset of $\emptyset \times Y$ is this function?" explains the terminology "empty function".)

(c) Show that the function in part (a) satisfies the definition of "injective function", hence (by definition) is injective.

(d) Show that empty function from the null set to itself satisfies the definition of "bijective function", hence (by definition) is a bijection. We will call this function $f : \emptyset \to \emptyset$ the "empty bijection".

Empty functions are not very interesting, of course. The main purpose of defining them is to enable various *other* definitions, theorems, etc., to be worded without an extra statement for the special case in which some set in the definition, theorem, etc., happens to be empty. (However, if the statement of a theorem allows for some function to be an empty function, then depending on what's being stated, it may be necessary to handle the empty-function and nonempty-function case separately in the proof.)

B2. Let X, Y, and Z be sets and let $f : X \to Y,\ g : Y \to Z$ be functions.

(a) Show that if f and g are bijective, then so is $g \circ f$. (If this problem is designated as a hand-in problem, hand in only the case in which X, Y, and Z are assumed nonempty.[1])

(b) Suppose $g \circ f$ is bijective. Does it follow that f and g are bijective? If your answer is "yes", prove it; if "no", give a counterexample.

[1]Based on the set-theoretic definition of "function", you should be able to figure out what subset of $X \times Z$ the composition $g \circ f : X \to Z$ is. From this, you should be able to show that (i) if f is an empty function, then so is $g \circ f$, and (ii) if g is an empty function, the $g \circ f$ exists if and only if f is the empty bijection (in which case $g \circ f$ is again an empty function, by (i)). Hence if f and g are bijections, then either $X = Y = Z = \emptyset$ and all three functions $f, g, g \circ f$ are the empty bijection, or all three sets X, Y, Z are nonempty.

B3. Define $f : \mathbf{R} \to \mathbf{R}^2$ by $f(x) = (\cos x, \sin x)$. Let

$$D = \{(x, y) \in \mathbf{R}^2 \mid x^2 + y^2 < 1\}$$

and

$$S = \{(x, y) \in \mathbf{R}^2 \mid x^2 + (y-1)^2 < 1\}.$$

Identify (a) $f^{-1}(S)$, (b) $f^{-1}(D)$, and (c) $f^{-1}(f((-\frac{\pi}{2}, \frac{\pi}{2})))$.

B4. Find a formula expressing the cardinality of the Cartesian product of two finite sets X, Y in terms of the cardinalities of X and Y. Justify your answer.

B5. Recall that the *power set* of a set X is the set $\mathcal{P}(X)$ of all subsets of X (an *element* of $\mathcal{P}(X)$ is a *subset* of X). For every set X, let 2^X denote the set of all functions from X to the two-element set $\{0, 1\}$.

(a) Let X be a set. Exhibit a bijection $\mathcal{P}(X) \to 2^X$ (or $2^X \to \mathcal{P}(X)$). Thus $\mathcal{P}(X)$ has the same cardinality as 2^X.

(b) If X is a finite nonempty set of cardinality n, what is the cardinality of 2^X? (Derive your formula; don't just write down the answer.) If your answer is correct, you will see where the notation "2^X" comes from.

In view of part (a), part (b) gives the cardinality of $\mathcal{P}(X)$ for any finite set.