MAA 4211, Fall 2018—Assignment 2's non-book problems

B1. Let X and Y be sets. Show that $X \sim$ (some subset of Y)—i.e. that there is a bijection from X to some subset of Y—if and only if there is an injective map $f : X \to Y$.

B2. Let X and Y be nonempty sets. Show that there exists an injective map $f : X \to Y$ if and only if there exists a surjective map $g : Y \to X$.

B3. Let X be a set. Show that the following are equivalent:

(i) X is countable.

(ii) $X \sim A$ for some $A \subset \mathbf{N}$.

(iii) There exists an injective map $f : X \to \mathbf{N}$.

B4. Prove that every infinite set has a countably infinite subset.

B5. Prove that a countable union of countable sets countable; i.e., if $\{A_i\}_{i\in I}$ is a collection of sets, indexed by $I \subset \mathbf{N}$, with each A_i countable, then $\bigcup_{i\in I} A_i$ is countable. *Hints*: (i) Show that it suffices to prove this for the case in which $I = \mathbf{N}$ and, for every $i \in \mathbf{N}$, the set A_i is nonempty. (This will simplify the rest of the argument, but is not an essential step.) (ii) In the case above, a result proven in class shows that for each $i \in \mathbf{N}$ there is a surjective map $f_i : \mathbf{N} \to A_i$. Use these maps to produce a surjective map $\mathbf{N} \times \mathbf{N} \to \bigcup_{i\in\mathbf{N}} A_i$, and then use earlier results (from class and/or homework) to conclude that $\bigcup_{i\in\mathbf{N}} A_i$ is countable.