MAA 4211, Fall 2018—Assignment 7's non-book problems

B1. (This problem gives a second proof of a lemma we proved in class.) Let (E, d) be a metric space, let $(p_n)_{n=1}^{\infty}$ be a convergent sequence in E, let $p = \lim_{n\to\infty} p_n$, let $N \in \mathbf{N}$, and let $\epsilon > 0$. Assume that for all $n, m \geq N$, $d(p_n, p_m) < \epsilon$. Use the appropriate half of the "sequential characterization of continuity" to prove that for all $n \geq N$, we have $d(p_n, p) \leq \epsilon$. *Hint*: Consider the function $f : E \to \mathbf{R}$ defined by $f(q) = d(p_n, q)$.

B2. Show by counterexample that in Rosenlicht's problem IV.3, if we remove the hypothesis that the sets S_1, S_2 are both closed, then we cannot conclude that f is continuous.

B3. In Rosenlicht problem III.6, you were asked to show that the set $S := \{(x, y) \in \mathbf{E}^2 \mid xy = 1, x > 0\}$ is closed in $\mathbf{E}^2$. The problem was quite difficult with the tools you had at that time. To help yourself appreciate the value of what we've accomplished in the last week, redo the problem (easily!) as follows:

(a) Let E, E' be metric spaces, $g : E \to E'$ a continuous function. Show that for any $q \in E'$, the set $f^{-1}(\{q\})$ is closed. (Remember that you have already done Rosenlicht problem IV.2.)

(b) Define $f : \mathbf{E}^2 \to \mathbf{R}$ by $f(x, y) = xy$. Since f is a polynomial in the coordinate functions on $\mathbf{E}^2$, f is continuous. Thus $f^{-1}(\{1\})$ is closed in $\mathbf{E}^2$. Show, however, that S is the intersection of the closed set $f^{-1}(\{1\})$ with another closed subset of $\mathbf{E}^2$ (hint: quadrants), hence is closed in $\mathbf{E}^2$.

B4. Redo Rosenlicht problem III.4 by expressing the set $\{(x, y) \in \mathbf{E}^2 \mid x > y\}$ as the inverse image of an open set under a continuous function $f : \mathbf{E}^2 \to \mathbf{R}$.

B5. (a) Show that in every metric space, every singleton set is connected. (Recall that a *singleton set* is a set with exactly one element.)

(b) A metric space E is called *totally disconnected* if the only nonempty connected subsets of E are the singleton sets. Recall that (E, d) is called *discrete* if for every $p \in E$ there exists $r > 0$ such that $B_r(p) = \{p\}$ (equivalently, if every point in E is isolated).[1] Show that every discrete metric space is totally disconnected.

(c) Prove that $\mathbf{Q}$, with its usual metric, is *not* discrete, but *is* totally disconnected.

B6. Let X, Y be metric spaces and let $f : X \to Y$ be a function.

(a) Show that if X is a *discrete* metric space, then $f : X \to Y$ is continuous. (Thus if X is discrete, *every* function from X to *any* metric space is continuous.)

[1] As has been mentioned in class several times, the metric in which the distance between any two distinct points is 1 is an *example* of a discrete metric; it is not "the" discrete metric.

(b) Show that if Y is a discrete metric space, and X is connected, then f is continuous if and only if f is a constant map.