MAA 4211

INTERIORS, CLOSURES, AND BOUNDARIES

updated October 8, 2018

The facts stated in problems 15-17 on p. 62 of Rosenlicht are of fundamental importance. Here is a summary of these and a few other facts; the proofs are exercises (some of which can be done in one line, and most of which can be done in three lines or less, if you do them in the order listed).

Below, (E, d) always represents an arbitrary metric space, $p \in E$ an arbitrary point, $S \subset E$ an arbitrary subset, and $\mathcal{C}S$ or $\mathcal{C}(S)$ the complement of S in E.

Definitions. Let $S \subset E$. A point p is an *interior point* of S if there exists an open ball centered at p entirely contained in S. The *interior of* S, written $\mathring{S}, S^\circ$, or $\text{Int}(S)$, is defined to be the set of interior points of S. The *closure of* S, written $\overline{S}$, is defined to be the intersection of all closed sets that contain S. The *boundary of* S, written ∂S, is defined by $\partial S = \overline{S} \cap \overline{\mathcal{C}S}$. (In other words, the boundary of a set is the intersection of the closure of the set and the closure of its complement.)

All the facts below are implicitly prefaced with "for all $S \subset E$".

Facts about interiors.

(1) $\mathring{S} \subset S$.

(2) $\mathring{S}$ is open.

(3) If $U \subset S$ is open, then $U \subset \mathring{S}$. (Because of this property and properties (1) and (2), $\mathring{S}$ is often referred to as "the largest open set contained in S".)

(4) S is open iff $S = \mathring{S}$.

(5) $\mathring{S}$ is the union of all open sets that are contained in S.

Facts about closures.

(6) $\overline{S}$ is the intersection of all closed sets that contain S (by definition).

(7) $\overline{S}$ is closed.

(8) $\overline{S} \supset S$.

(9) If $U \supset S$ is closed, then $U \supset \overline{S}$. (Because of this property and properties (7) and (8) , $\overline{S}$ is often referred to as "the smallest closed set containing S".)

(10) S is closed iff $S = \overline{S}$.

(11) $\overline{S} = \mathcal{C}(\text{Int}(\mathcal{C}S))$; equivalently, $\mathcal{C}\overline{S} = \text{Int}(\mathcal{C}S)$. (In words: the complement of the closure is the interior of the complement.)

(12) $p \in \overline{S}$ iff every open ball centered at p contains a point of S.

(13) $\overline{S}$ is the set of all limits of E-convergent sequences of points in S. (Here "E-convergent" means "convergent in (E, d)".)

Facts about boundaries.

(14) $\partial(\mathcal{C}S) = \partial S$.

(15) $\partial S \subset \overline{S}$.

(16) $p \in \partial S$ iff every open ball centered at p contains a point of S and a point of $\mathcal{C}S$.

(17) $E = S^\circ \amalg \partial S \amalg (\mathcal{C}S)^\circ$. (The disjoint-union symbol $\amalg$ means the almost the same thing as "union", but is applied only to disjoint sets; the statement "$X = Y \amalg Z$" means that Y and Z are disjoint sets whose union is X. Similarly, "$X = Y \amalg Z \amalg W$" means that Y, Z and W are *pairwise* disjoint sets—no two of these sets intersect—whose union is X.[1]) Remark: everything below is an almost immediate consequence of "$E = S^\circ \amalg \partial S \amalg (\mathcal{C}S)^\circ$". However, this is not the only way to prove these facts.

(18) $\mathring{S} = S \setminus \partial S = \overline{S} \setminus \partial S$. (Recall that $A \setminus B$ denotes the set of points lying in A but not B; equivalently, $A \setminus B = A \cap \mathcal{C}B$.)

(19) $\overline{S} = \mathring{S} \amalg \partial S$.

(20) $\overline{S} = S \cup \partial S$.

(21) S is closed iff $S \supset \partial S$.

(22) S is open iff S and ∂S are disjoint.

[1] More generally, the notation "$X = \amalg_{\alpha \in A} Y_\alpha$" means that $\{Y_\alpha : \alpha \in A\}$ is a collection of pairwise disjoint sets, indexed by the set A, and that X is the union of these sets.