MAA 4211, Fall 2019—Assignment 1's non-book problems

B1. Let X, Y, and Z be sets and let $f : X \to Y,\ g : Y \to Z$ be functions.

(a) Show that if f and g are bijective, then so is $g \circ f$. (If this problem is designated as a hand-in problem, hand in only the case in which X, Y, and Z are assumed nonempty.[1])

(b) Suppose $g \circ f$ is bijective. Does it follow that f and g are bijective? If your answer is "yes", prove it; if "no", give a counterexample.

B2. Define $f : \mathbf{R} \to \mathbf{R}^2$ by $f(x) = (\cos x, \sin x)$. Let

$$D = \{(x, y) \in \mathbf{R}^2 \mid x^2 + y^2 < 1\}$$

and

$$S = \{(x, y) \in \mathbf{R}^2 \mid x^2 + (y-1)^2 < 1\}.$$

Identify (a) $f^{-1}(S)$, (b) $f^{-1}(D)$, and (c) $f^{-1}(f((-\frac{\pi}{2}, \frac{\pi}{2})))$.

Note: In part (c), you are responsible for correctly determining the meaning of the notation "$(-\frac{\pi}{2}, \frac{\pi}{2})$". In general, given real numbers a, b with $a < b$, there are two very different objects that the notation "(a, b)" could represent: a point in $\mathbf{R}^2$, or an open interval in $\mathbf{R}$. In part (c), only one of these two meanings makes any sense in the expression "$f((-\frac{\pi}{2}, \frac{\pi}{2}))$", and therefore that meaning is the intended one.[2]

B3. Find a formula expressing the cardinality of the Cartesian product of two finite sets X, Y in terms of the cardinalities of X and Y. Justify your answer.

B4. Recall that the *power set* of a set X is the set $\mathcal{P}(X)$ of all subsets of X (an *element* of $\mathcal{P}(X)$ is a *subset* of X). For every set X, let 2^X denote the set of all functions from X to the two-element set $\{0, 1\}$.

(a) Let X be a set. Exhibit a bijection from $\mathcal{P}(X)$ to 2^X (or from 2^X to $\mathcal{P}(X)$). Thus $\mathcal{P}(X)$ has the same cardinality as 2^X.

(b) If X is a finite nonempty set of cardinality n, what is the cardinality of 2^X? (Derive your formula; don't just write down the answer.) If your answer is correct, you will see where the notation "2^X" comes from.

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[1] Based on the set-theoretic definition of "function", you should be able to figure out what subset of $X \times Z$ the composition $g \circ f : X \to Z$ is. From this, you should be able to show that (i) if f is an empty function, then so is $g \circ f$, and (ii) if g is an empty function, the $g \circ f$ exists if and only if f is the empty bijection (in which case $g \circ f$ is again an empty function, by (i)). Hence if f and g are bijections, then either $X = Y = Z = \emptyset$ and all three functions $f, g, g \circ f$ are the empty bijection, or all three sets X, Y, Z are nonempty.

[2] There are even more meanings for the notation "(a, b)" in other contexts. For example, in number theory, if a and b are positive integers then "(a, b)" denotes their greatest common divisor.

In view of part (a), the answer to part (b) also gives the cardinality of $\mathcal{P}(X)$ for any finite set.

B5. Let X and Y be sets. Show that $X \sim$ (some subset of Y)—i.e. that there is a bijection from X to some subset of Y—if and only if there is an injective map $f : X \to Y$.

B6. Let X and Y be nonempty sets. Show that there exists an injective map $f : X \to Y$ if and only if there exists a surjective map $g : Y \to X$.