MAA 4211, Fall 2019—Assignment 2's non-book problems

B1. Let X and Y be sets. Show that $X \sim$ (some subset of Y)—i.e. that there is a bijection from X to some subset of Y—if and only if there is an injective map $f : X \to Y$.

B2. Let X and Y be nonempty sets. Show that there exists an injective map $f : X \to Y$ if and only if there exists a surjective map $g : Y \to X$.

B3. Let X be a set. Show that the following are equivalent:

(i) X is countable.

(ii) $X \sim A$ for some $A \subset \mathbf{N}$.

(iii) There exists an injective map $f : X \to \mathbf{N}$.

Note added after original posting: The equivalence "(i) $\iff$ (ii)" isn't something you have to *show*; it's true by our definition of "countable". When I first wrote this problem (years ago), the definition of countable that I'd given my students was "X is countable if either X is finite or $X \sim \mathbf{N}$." This older definition reflects the origin of the term "countable" better than the definition I gave you, but is less convenient to use.

B4. Prove that every infinite set has a countably infinite subset.

B5. Prove that a countable union of countable sets countable; i.e., if $\{A_i\}_{i\in I}$ is a collection of sets, indexed by $I \subset \mathbf{N}$, with each A_i countable, then $\bigcup_{i\in I} A_i$ is countable. *Hints*: (i) Show that it suffices to prove this for the case in which $I = \mathbf{N}$ and, for every $i \in \mathbf{N}$, the set A_i is nonempty. (This will simplify the rest of the argument, but is not an essential step.) (ii) In the case above, a result proven in class shows that for each $i \in \mathbf{N}$ there is a surjective map $f_i : \mathbf{N} \to A_i$. Use these maps to produce a surjective map $\mathbf{N} \times \mathbf{N} \to \bigcup_{i\in\mathbf{N}} A_i$, and then use earlier results (from class and/or homework) to conclude that $\bigcup_{i\in\mathbf{N}} A_i$ is countable.

B6. Let $(\mathbf{F}, <)$ be an ordered field, let S be a nonempty subset of $\mathbf{F}$, let $c \in \mathbf{F}$, and **for purposes of this problem** let $cS = \{cx \mid x \in S\}$. (Do *not* use this notation outside this problem without defining what you mean by the notation.)

(a) Assume that $c > 0$.

   (i) Show that an element $b \in \mathbf{F}$ is an upper bound for S if and only if cb is an upper bound for cS.

   (ii) Repeat part (i) with "upper bound" replaced by "lower bound".

(iii) Show that S has a least upper bound if and only if cS has a least upper bound, and that (when these sets have least upper bounds), they are related by

$$\text{l.u.b.}(cS) = c \cdot \text{l.u.b.}(S).$$

(If you're wondering, "Can we assume that $(\mathbf{F}, <)$ has the Least Upper Bound Property?" you should be able to answer your own question by looking at the statement of the problem.)

(iv) Repeat part (iii) with "least upper bound" replaced by "greatest lower bound".

(b) Assume that $c < 0$. Figure out, and prove, the correct "if and only if" relations between upper/lower bounds of S and those of cS, and between least-upper/greatest-lower bounds of S and those of cS.

B7. We say that an ordered field $(\mathbf{F}, <)$ has the *Greatest Lower Bound Property* if every nonempty subset that bounded from below has a greatest lower bound. Show that $(\mathbf{F}, <)$ has the Greatest Lower Bound Property if and only if $(\mathbf{F}, <)$ has the Least Upper Bound Property.

(The argument is sketched on p. 25 of Rosenlicht. What you're being asked to do here is to justify statements that were *asserted* in Rosenlicht without explicit justification. One easy way to do this is to use certain results from problem B6.)

B8. Let $(\mathbf{F}, <)$ be an ordered field, let $a \in \mathbf{F}_+$, and let $S = \{x \in \mathbf{F}_+ \mid x^2 < a\}$. By appropriately modifying parts of the proof given in class that every positive *real* number has a real square root, show the following:

(a) If $x \in \mathbf{F}_+$ and $x^2 < a$ (i.e. if $x \in S$), then x is *not* an upper bound for S.

(b) If $x \in \mathbf{F}_+$ and $x^2 \geq a$, then x *is* an upper bound for S.

(c) If $x \in \mathbf{F}_+$ and $x^2 > a$, then there exists $\delta \in \mathbf{F}_+$ such that $x - \delta$ is an upper bound for S (and hence x is not a *least* upper bound of S).