MAA 4211, Fall 2019—Assignment 3's non-book problems

B1. (a) Let Z be any nonempty set, and let $\mathrm{Func}(Z,\mathbf{R})$ denote the set of all functions $Z \to \mathbf{R}$. As temporary notation, just for this problem, let $\underline{\mathbf{0}}$ denote the constant function with value 0 (i.e. $\underline{\mathbf{0}}(z) = 0$ for all $z \in Z$). For $f, g \in \mathrm{Func}(Z,\mathbf{R})$ and $c \in \mathbf{R}$ we define $f + g \in \mathrm{Func}(Z,\mathbf{R})$ and $cf \in \mathrm{Func}(Z,\mathbf{R})$ by

$$\begin{aligned} f + g &= \text{the function } z \mapsto f(z) + g(z), \\ cf &= \text{the function } z \mapsto cf(z). \end{aligned}$$

Check that, with the operations above, $\mathrm{Func}(Z,\mathbf{R})$ is a vector space with zero-element $\underline{\mathbf{0}}$.

(b) Let $\mathbf{R}^\infty$ denote the set of all functions $\mathbf{N} \to \mathbf{R}$. For $f \in \mathbf{R}^\infty$, one of the notations we commonly use is $(x_1, x_2, x_3, \ldots)$, where for each $n \in \mathbf{N}$ the number x_n is $f(n)$. Thus an element of $\mathbf{R}^\infty$ is also called an *infinite sequence* in $\mathbf{R}$. By part (a), $\mathbf{R}^\infty$ is a vector space. Check that, in the sequence-notation above, the operations and zero-element in $\mathbf{R}^\infty$ (as defined in part (a)) are given by

$$\begin{aligned} (x_1, x_2, x_3, \ldots) + (y_1, y_2, y_3, \ldots) &= (x_1 + y_1, x_2 + y_2, x_3 + y_3, \ldots), \\ c(x_1, x_2, x_3, \ldots) &= (cx_1, cx_2, cx_3, \ldots), \\ \underline{\mathbf{0}} = \vec{0} &:= (0, 0, 0, \ldots). \end{aligned}$$

(c) A real-valued sequence $\vec{x} = (x_1, x_2, x_3, \ldots) \in \mathbf{R}^\infty$ (switching from the notation "f" in part (b)) is called *bounded* if the set $\{|x_n| : n \in \mathbf{N}\}$ is bounded; equivalently, if there exists $M \in \mathbf{R}$ such that for all $n \in \mathbf{N}$ we have $|x_n| \leq M$. Let $\mathbf{R}^\infty_b \subset \mathbf{R}^\infty$ denote the set of bounded real-valued sequences. Show that $\mathbf{R}^\infty_b$ is a vector subspace of $\mathbf{R}^\infty$.

(d) For any $\vec{x} \in \mathbf{R}^\infty_b$, the set $\{|x_n| : n \in \mathbf{N}\}$ is nonempty and bounded above, hence has a least upper bound. Therefore we can define a function $\|\ \|_\infty : \mathbf{R}^\infty_b \to \mathbf{R}$ by

$$\|\vec{x}\|_\infty := \sup\{|x_n| : n \in \mathbf{N}\}.$$

(Here, "sup", which stands for "supremum", is synonymous with "least upper bound", and is actually the more commonly used term except when the focus is on properties of the real-number system.[1] It's pronounced like "soup".) Show that $\|\ \|_\infty$ is a norm on the vector space $\mathbf{R}^\infty_b$. (Note: the "∞" subscript in $\|\ \|_\infty$ has nothing to do with the "∞" superscript in $\mathbf{R}^\infty$. Rather, the notation for this norm comes from the analogous norm on $\mathbf{R}^n$, where we can replace "sup" by "max".) We call this norm the ℓ^∞*-norm* or *sup-norm* on $\mathbf{R}^\infty_b$.

(e) Let d_∞ denote the metric on $\mathbf{R}^\infty_b$ associated with the ℓ^∞ norm; we call d_∞ the ℓ^∞ *metric* on $\mathbf{R}^\infty_b$. Check that d_∞ is given by

$$d_\infty(\vec{x}, \vec{y}) = \sup\{|x_n - y_n| : n \in \mathbf{N}\}.$$

[1] The concept of *supremum* is actually more general than "least upper bound", but reduces to "least upper bound" for subsets of $\mathbf{R}$ that are bounded above. We may discuss the more general usage later in the course.

B2. Let $(V, \| \, \|)$ be a normed vector space, viewed as a metric space with the associated metric. Show that for all $v \in V$,

$$\text{for each } r > 0 \text{ we have} \quad B_r(v) \;=\; \{v + w \mid w \in B_r(0)\},$$

and

$$\text{for each } r \geq 0 \text{ we have} \quad \overline{B}_r(v) \;=\; \{v + w \mid w \in \overline{B}_r(0)\}.$$

In other words, each open (respectively, closed) ball centered at a given v is simply the translation, by v, of the open (respectively, closed) ball of the same radius centered at the origin.

> Note: (1) In each of the displayed statements above, the symbol "0" has two different meanings. You are expected to be able to tell, from context, what each meaning is. (2) Usually, allowing the same notation to have two different meanings in the same sentence (or paragraph, proof, etc.) is a terrible idea, deserving of a bad-writing penalty. The multiple-meanings use of "0" is an exception to this rule, and you'll find "0" used this way by most mathematicians and in most textbooks. One reason is that there's a zero element of every field, every vector space, and, more generally, every abelian group[2]; using different notation for each zero-element can lead to hard-to-read clutter. Another reason for making this exception is that "0" isn't usually a symbol you *introduce*; you treat it as having already been introduced, for every algebraic structure that has an element called "zero", prior to your having started writing. Nonetheless, *sometimes*, as in problem B1, using different notation for different zero-elements is nearly essential to prevent confusion.

B3. Let (E, d) be a metric space and let $X \subset E$ be a nonempty subset. For $r > 0$ and $p \in X$, let $B_r^E(p)$ and $B_r^X(p)$ denote the open balls of radius r and center p in the metric spaces (E, d) and $(X, d|_X)$ respectively. (As stated in class, '$d|_X$' is "abuse of notation" that we're allowing for '$d|_{X \times X}$'.) Similarly, let $\overline{B}_r^E(p)$ and $\overline{B}_r^X(p)$ denote the closed balls of radius r and center p in the metric spaces indicated by the superscripts. Show that, for all such r and p,

$$\begin{aligned} B_r^X(p) &= B_r^E(p) \bigcap X \\ \text{and} \quad \overline{B}_r^X(p) &= \overline{B}_r^E(p) \bigcap X. \end{aligned}$$

B4. Let (E, d) be a metric space, let $X \subset E$ be a nonempty subset. For purposes of this problem, let us refer to the open (respectively, closed) subsets of (E, d) as being *E-open* (respectively, *E-closed*), and refer to the open (respectively, closed) subsets of $(X, d|_X)$ as being *X-open* (respectively, *X-closed*).

[2]Ignore the last statement if you don't know what an *abelian group* is. It's something you'd learn about in a course in abstract algebra, which is not a prerequisite for this class, and is not a concept we'll be using.

(a) Show that a subset $U \subset X$ is X-open if and only if there is an E-open subset W such that $U = X \bigcap W$.

Hint: Use problem B3 and the propositions on pp. 39–40 of Rosenlicht. (These propositions were combined into what was called "Proposition 1" in class on Wednesday 10/2/19.) You will not need *all* of the proposition on p. 39, just part of it.

(b) Show that the analogous statement is true with "open" replaced by "closed": a subset $U \subset X$ is X-closed if and only if there is an E-closed subset W such that $U = X \bigcap W$.

B5. Define a metric d on the set of rational numbers $\mathbf{Q}$ by $d(x, y) = |x - y|$ (the "standard metric on $\mathbf{Q}$", simply the restriction to $\mathbf{Q}$ of the standard metric on $\mathbf{R}$). Give an example, with proof, of a nonempty, proper subset of $(\mathbf{Q}, d)$ that is both open and closed in this metric space. (Do not expect your subset to be either open or closed in $\mathbf{R}$, let alone *both* open and closed in $\mathbf{R}$. As we will see in a few weeks, there is no nonempty, proper subset of $\mathbf{R}$ that is both open and closed with respect to the standard metric.)

B6. Let $n \geq 1$ and let $\mathbf{E}^n$ denote Euclidean n-space. Let $p \in \mathbf{E}^n$, $r \geq 0$. Prove that the closed ball $\overline{B}_r(p)$ is not an open set.

Remember: (i) "Closed" does not imply "not open". The fact that a closed ball is a closed set doesn't imply that a closed ball can't also be an open set. (In fact, in one of the Rosenlicht problems you will see an example in which *every* ball is simultaneously an open set and a closed set.) (ii) There is no such thing as "proof by picture". If you are asserting, for example, that a certain open ball contains points of some other set, you have to *prove* that assertion, not merely assert that it's true because it looks that way in a picture you've drawn. (That's an instance of "proof by lack of imagination": you believe that some fact is true simply because you can't think of an example in which that fact would be false, and then assert that that fact *is* true without supplying any logical argument, e.g. proof by contradiction, that relies only on the given hypotheses and previously proven facts.)

B7. Let (E, d) be a metric space. For purposes of this problem, for each $p \in E$ define a property we'll call "boundedness with respect to p" as follows: a set $S \subset E$ is *bounded with respect to* p if S is contained in some ball *centered at* p.

Let $p \in E$. Show that for every $S \subset E$, the following are equivalent:

(i) S is bounded with respect to p.

(ii) S is bounded.

(iii) S is bounded with respect to q for every $q \in E$.

B8. Let $(E, d) = \mathbf{E}^2$ (Euclidean 2-space). Let $p \in E$ and let $r > 0$.

(a) Show that $\overline{B_r(p)} = \overline{B}_r(p)$ (i.e. the closure of an open ball is the closed ball with the same center and radius). *Note*: This is not true in every metric space! See problems B9 and B10.

(b) Show that $\partial B_r(p)$ is the *sphere* of radius r centered at p, defined as $\{q \in E \mid d(p, q) = r\}$. (This is the general definition of "sphere" for an arbitrary metric space; spheres in $\mathbf{E}^2$ are circles.) *Note*: This is also not true in every metric space!

(c) Re-do parts (a) and (b) with $\mathbf{E}^2$ replaced by an arbitrary normed vector space $(V, \| \ \|)$. Once (a) and (b) are done, you should find this easy; if not, then your arguments in (a) and (b) are probably wrong.

B9. Let (E, d) be a metric space, $p \in E$, and $r > 0$. Let $S_r(p)$ denote the sphere of radius r centered at p.

(a) Prove that $\partial(B_r(p)) \subset S_r(p)$.

(b) Prove that $\overline{B}_r(p)) = \overline{B_r(p)}$ if and only if $\partial(B_r(p)) = S_r(p)$.

B10. Give an example of a metric space E in which there is an open ball $B_r(p)$ whose closure is *not* the closed ball $\overline{B}_r(p)$. (You have already encountered a metric space with this property.)

Note that in view of problem B9(b), for any such ball we have $\partial(B_r(p)) \neq S_r(p)$.