MAA 4211, Fall 2019—Assignment 4's non-book problems

B1. Let (E, d) be a metric space, let $(p_n)_{n=1}^{\infty}$ be a sequence in E, and define sequences $(x_n)_{n=1}^{\infty}$ and $(y_n)_{n=1}^{\infty}$ by

$$\begin{aligned} x_n &= p_{2n-1} \text{ for each } n \in \mathbf{N}, \\ y_n &= p_{2n} \text{ for each } n \in \mathbf{N}. \end{aligned}$$

(In other words, (x_n) and (y_n) are the subsequences of (p_n) given by the odd-numbered terms and even-numbered terms, respectively.) Prove that the following are equivalent:

(i) $(p_n)_{n=1}^{\infty}$ converges.

(ii) Both $(x_n)_{n=1}^{\infty}$ and $(y_n)_{n=1}^{\infty}$ converge, and their limits are equal.

Prove also that if condition (ii) holds, then $\lim_{n\to\infty} p_n = \lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n$.

B2. Let d_1 and d_2 be two metrics on a nonempty set E.[1] Call a set $S \subset E$ "d_1-open" if it is open in the metric space (E, d_1), and "d_2-open" if it is open in the metric space (E, d_2). Analogously define "d_i-bounded set", "d_i-convergent sequence", and, for a sequence in E and point $q \in E$, the property "d_i-convergent to q".

(a) Suppose that there exists $c > 0$ such that $d_2(p, q) \le c d_1(p, q)$ for all $p, q \in E$. Prove that every d_2-open subset of E is d_1-open.

Note: If there exists *any* $c \in \mathbf{R}$ such that $d_2(p, q) \le c d_1(p, q)$ for all $p, q \in E$, then there exists a positive such c:

- If E contains at least two points, say p_1 and p_2, then $d_1(p_1, p_2)$ and $d_2(p_1, p_2)$ are both positive, so if $d_2(p_1, p_2) \le c d_1(p_1, p_2)$ then c must be positive as well.
- If E contains exactly one point p, then $d_2(p, p) = 0 \le c \cdot 0 = c d_1(p, p)$ for *every* $c \in \mathbf{R}$, so, in particular, any positive c works.

(b) Metrics d_1, d_2 on a set E are called *equivalent* if there exist $c_1, c_2 > 0$ such that for all $p, q \in E$, $d_2(p, q) \le c_1 d_1(p, q)$ and $d_1(p, q) \le c_2 d_2(p, q)$.[2] Below, we write "$d_1 \sim d_2$" for "$d_1, d_2$ are equivalent metrics" (on a given set).

Let E be an arbitrary nonempty set. Prove the following:

[1] Everything you'll be proving in this problem is true even if E is empty; I'm just not asking you to spend time to deal with this trivial case, or to make sure you word your arguments in such a way that they apply equally well whether or not E is empty.

[2] See the note in part (a). If there exist *any* $c_1, c_2 \in \mathbf{R}$ for which the preceding condition holds, then there exist *positive* c_1 and c_2 for which the condition holds. Thus, the requirement in this definition that c_1, c_2 be positive is redundant. It has been put into the definition explicitly just so that in proving assertions about equivalent metrics, you don't have to spend time showing that you can take c_1 and c_2 to be positive.

(i) The relation $\sim$ is an equivalence relation on the set of all metrics on E.

(ii) Equivalent metrics on E determine the same open sets and the same closed sets. I.e. if d_1 and d_2 are equivalent and $U \subset E$, then U is d_1-open iff U is d_2-open, and U is d_1-closed iff U is d_2-closed.

(iii) Equivalent metrics determine the same bounded sets. I.e. if d_1 and d_2 are equivalent and $U \subset E$, then U is d_1-bounded if U is d_2-bounded.

(iv) Equivalent metrics determine the same convergent sequences and the same limits of convergent sequences. I.e. if d_1 and d_2 are equivalent, $q \in E$, and $(p_n)_{n=1}^{\infty}$ is a sequence in E, then (p_n) is d_1-convergent to q iff (p_n) is d_2-convergent to q (hence (p_n) is d_1-convergent iff (p_n) is d_2-convergent).

B3. Let $\| \ \|$ and $\| \ \|'$ be two norms on a vector space V. We call these two norms *equivalent* if there exist $c_1, c_2 > 0$ such that for all $v \in V$, $\|v\| \leq c_1\|v\|'$ and $\|v\|' \leq c_2\|v\|$. [3]

(a) Prove that if norms $\| \ \|$ and $\| \ \|'$ are equivalent, then their associated metrics are equivalent.

(b) Check that "equivalence of norms" is an equivalence relation on the set of all norms on V. To do this, simply look back at your proof of B2(b)(i), and check in your head that the same argument works if you replace all your expressions of the form "$d(p,q)$" with expressions of the form "$\|v\|$".

B4. Prove that, for each $n \in \mathbf{N}$, the ℓ^1, ℓ^2, and ℓ^∞ norms on $\mathbf{R}^n$ are all equivalent to each other (i.e. each is equivalent to the other two), and hence that their associated metrics are equivalent to each other. Note that problem B3(b) can be used to reduce from three to two the number of norm-pair comparisons you need to do.

B5. Let (E,d) be a metric space, let $(p_n)_{n=1}^{\infty}$ be a Cauchy sequence in E, and assume that this sequence has a convergent subsequence $(p_{n_i})_{i=1}^{\infty}$. Let $p = \lim_{i\to\infty} p_{n_i}$. Show that the original sequence $(p_n)_{n=1}^{\infty}$ also converges to p.

(Note (E,d) is not assumed to have any properties other than being a metric space; e.g. we are not assuming (E,d) is complete. The hypotheses say only that *this particular* Cauchy sequence $(p_n)_{n=1}^{\infty}$ has a convergent subsequence, not that every sequence has a convergent subsequence, and not that every Cauchy sequence has a convergent subsequence.)

B6. (This problem is essentially a continuation of B2b. We had not yet defined all the terminology in this problem when B2 was posted.)

[3] Some mathematicians use the term "strongly equivalent" for *metrics* related to each other as in problem B2. What these mathematicians call "equivalent metrics" is what I call "*topologically* equivalent metrics". However, for *norms*, there is universal agreement on the terminology "equivalent".

Let d_1 and d_2 be two metrics on a nonempty set E. For $i \in \{1,2\}$, call a sequence d_i-*Cauchy* if it is Cauchy in (E, d_i).

(a) Prove that a sequence in E is d_1-Cauchy iff the sequence is d_2-Cauchy.

(b) Prove that (E, d_1) is complete iff (E, d_2) is complete.

B7. Let (E_1, d_1) and (E_2, d_2) be metric spaces. In earlier homework (Rosenlicht, p. 61/1c) you showed that the function $d : (E_1 \times E_2) \times (E_1 \times E_2) \to \mathbf{R}$ defined by

$$d((x_1, x_2), (y_1, y_2)) = \max\{d_1(x_1, y_1), d_2(x_2, y_2)\}$$

is a metric on $E_1 \times E_2$.

(a) Show that the function $d' : (E_1 \times E_2) \times (E_1 \times E_2) \to \mathbf{R}$ defined by

$$d'((x_1, x_2), (y_1, y_2)) = d_1(x_1, y_1) + d_2(x_2, y_2),$$

is also a metric on $E_1 \times E_2$.

(b) Show that the metrics d and d' on $E_1 \times E_2$ are equivalent. (Reminder: "Show" always means "Prove".)

(c) Show that if (E_1, d_1) and (E_2, d_2) are complete, then so are $(E_1 \times E_2, d)$ and $(E_1 \times E_2, d')$.

(d) Repeat parts (a), (b), and (c) for the function $d'' : (E_1 \times E_2) \times (E_1 \times E_2) \to \mathbf{R}$ defined by

$$d''((x_1, x_2), (y_1, y_2)) = \sqrt{d_1(x_1, y_1)^2 + d_2(x_2, y_2)^2}\ .$$