MAA 4211, Fall 2019—Assignment 5's non-book problems

B1. (Strengthening of Rosenlicht problem III.10.) Let $(p_n)_{n=1}^{\infty}$ be a convergent sequence in a metric space (E,d), let R be the range of this sequence, and let $p = \lim_{n\to\infty} p_n$. Show that $R \bigcup \{p\}$ is a compact subset of E.

B2. Let (E,d) be a metric space and let $S \subset E$.

(a) Prove that $\overline{S} = S \bigcup \{\text{all cluster points of } S\}$.

(b) Prove that $\overline{S} = S \bigcup \{\text{all cluster points of } S \text{ that lie in } \partial S\}$.

(c) Prove this nearly-trivial corollary of part (a): S is closed if and only if S contains all its cluster points.

B3. Let d_1, d_2 be equivalent metrics on a set E. Without using any relations between compactness and sequential compactness, (none of which we've established as of the date this problem is being posted), prove that (E,d_1) is sequentially compact if and only if (E,d_2) is sequentially compact.

B4. As in previous homework, let $\mathbf{R}^{\infty}$ denote the vector space whose elements are real-valued sequences, and let $\mathbf{R}^{\infty}_b \subset \mathbf{R}^{\infty}$ denote the space of bounded real-valued sequences. We will write many elements of $\mathbf{R}^{\infty}$ using the notation $\vec{a}$ to stand for $(a_i)_{i=1}^{\infty}$, the notation $\vec{b}$ to stand for $(b_i)_{i=1}^{\infty}$, etc. (Thus if we are given an element $\vec{a} \in \mathbf{R}^{\infty}$, it is understood that the notation "a_i" means the i^{th} term of $\vec{a}$.) We often think of elements of $\mathbf{R}^{\infty}$ as "infinitely long row-vectors" (i.e. vectors with infinitely many components), and think of the i^{th} term of an element of $\mathbf{R}^{\infty}$ as the i^{th} component of this vector.

The normed vector space $(\mathbf{R}^{\infty}_b, \|\ \|_{\infty})$ is conventionally called $\ell^{\infty}(\mathbf{R})$. As with any normed vector space, when we speak of metric-space properties of $\ell^{\infty}(\mathbf{R})$, the metric is assumed to be the one associated with the given norm (unless otherwise specified); thus the ℓ^{∞} metric on $\mathbf{R}^{\infty}_b$ is the function $d : \mathbf{R}^{\infty}_b \times \mathbf{R}^{\infty}_b \to \mathbf{R}$ given by $d(\vec{a},\vec{b}) = d_{\infty}(\vec{a},\vec{b}) = \sup\{|a_i - b_i| \ : \ i \in \mathbf{N}\}$ (exactly the metric in Rosenlicht p. 61/#1b).

Since a sequence in $\ell^{\infty}(\mathbf{R})$ is a sequence of sequences, to avoid confusion in this problem we will use a superscript rather than a subscript to label the terms of a sequence in $\ell^{\infty}(\mathbf{R})$; we will write such a sequence as $(\vec{a}^{(n)})_{n=1}^{\infty}$. Thus the n^{th} term in such a sequence is a real-valued sequence $\vec{a}^{(n)} = (a_i^{(n)})_{i=1}^{\infty} = (a_1^{(n)}, a_2^{(n)}, a_3^{(n)}, \dots)$. You may find it helpful to picture such a sequence as an array with infinitely many rows and columns, in which the first row is the sequence $\vec{a}^{(1)}$, the second row is the sequence $\vec{a}^{(2)}$, etc.:

$$\begin{array}{ccccc} a_1^{(1)} & a_2^{(1)} & a_3^{(1)} & a_4^{(1)} & \dots \\ a_1^{(2)} & a_2^{(2)} & a_3^{(2)} & a_4^{(2)} & \dots \\ a_1^{(3)} & a_2^{(3)} & a_3^{(3)} & a_4^{(3)} & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{array}$$

(a) Let $(\vec{a}^{(n)})_{n=1}^{\infty}$ be a Cauchy sequence in $\ell^\infty(\mathbf{R})$. Show that for all $i \in \mathbf{N}$, the real-valued sequence $(a_i^{(n)})_{n=1}^{\infty}$ (the sequence of "i^{th} components"—more precisely, i^{th} terms—of the $\vec{a}^{(n)}$, corresponding to the i^{th} column in the diagram above) is a Cauchy sequence in $\mathbf{R}$. Note that in $(a_i^{(n)})_{n=1}^{\infty}$, the index i is fixed; it is n that varies: $(a_i^{(n)})_{n=1}^{\infty} = (a_i^{(1)}, a_i^{(2)}, a_i^{(3)}, \dots)$.

(b) Let $(\vec{a}^{(n)})_{n=1}^{\infty}$ be as in part (a). Since $\mathbf{R}$ is complete, for all $i \in \mathbf{N}$ there exists $c_i \in \mathbf{R}$ such that $\lim_{n\to\infty} a_i^{(n)} = c_i$. Let $\vec{c}$ be the sequence $(c_i)_{i=1}^{\infty} \in \mathbf{R}^\infty$—no subscript "$b$", yet. Show that the sequence $\vec{c}$ *is*, in fact, bounded. (So $\vec{c} \in \mathbf{R}_b^\infty$ after all.)

(c) Let $(\vec{a}^{(n)})_{n=1}^{\infty}$ and $\vec{c}$ be as in part (b). Show that $(\vec{a}^{(n)})_{n=1}^{\infty}$ converges in $\ell^\infty(\mathbf{R})$ to $\vec{c}$. (Note: unlike for sequences in $\mathbf{R}^m$, this CANNOT be deduced just from the fact that $(a_i^{(n)})_{n=1}^{\infty}$ converges to c_i for all i; see part (e) below.) Thus $\ell^\infty(\mathbf{R})$ is complete.

Note: As of the date part (c) is being posted (11/14/19), you have not yet seen a lemma that is *extremely* useful in doing this problem-part. (**Update: This lemma was sent to you by email on 11/18/19.**)

Hint: For $\epsilon > 0$, if $N \in \mathbf{N}$ is as in the Cauchy criterion for the sequence $(\vec{a}^{(n)})_{n=1}^{\infty}$ in $\ell^\infty(\mathbf{R})$, show that for all $i \in \mathbf{N}$, this same N "works" in the Cauchy criterion for the real-valued sequence $(a_i^{(n)})_{n=1}^{\infty}$. (You probably already did this in part (b).) Then apply the aforementioned lemma to each sequence $(a_i^{(n)})_{n=1}^{\infty}$.

Notation for the remaining parts of this problem. For $n \in \mathbf{N}$, let $\vec{e}^{(n)} \in \mathbf{R}_b^\infty$ be the sequence whose n^{th} term is 1 and all of whose other terms are zero (e.g. $\vec{e}^{(3)} = (0,0,1,0,0,0,0,\dots)$).

(d) Show that for all $i \in \mathbf{N}$, the real-valued sequence $(e_i^{(n)})_{n=1}^{\infty}$ converges in $\mathbf{R}$ to 0.

(e) Let $\vec{0}$ be the zero element of $\mathbf{R}_b^\infty$ (the sequence $(0,0,0,0\dots)$). Compute $d(\vec{e}^{(n)}, \vec{0})$ for all n, and use your answer to show that $(\vec{e}^{(n)})_{n=1}^{\infty}$ does not converge in $\ell^\infty(\mathbf{R})$ to $\vec{0}$, even though the i^{th}-component sequence $(e_i^{(n)})_{n=1}^{\infty}$ converges to the i^{th} component of $\vec{0}$ for every i.

(f) Compute $d(\vec{e}^{(n)}, \vec{e}^{(m)})$ for all $m, n \in \mathbf{N}$ with $m \neq n$. Use your answer to show that no subsequence of $(\vec{e}^{(n)})_{n=1}^{\infty}$ can be Cauchy. Use this to deduce that no subsequence of $(\vec{e}^{(n)})_{n=1}^{\infty}$ can converge.

Note: since any sequence is trivially a subsequence of itself, the last conclusion implies that $(\vec{e}^{(n)})_{n=1}^{\infty}$ does not converge in $\ell^\infty(\mathbf{R})$ to *anything*, so, in particular, it does not converge to $\vec{0}$. But I still want you to do part (e) by the method indicated in part (e).)

(g) Use part (f) to deduce that the closed unit ball $\bar{B}_1(\vec{0}) \subset \ell^\infty(\mathbf{R})$ is not sequentially compact. (Thus this ball is a closed, bounded subset of a complete normed vector space, but is not compact. The Heine-Borel Theorem is false in infinite dimensions.

More precisely, in statement of the Heine-Borel Theorem, if we replace $\mathbf{E}^n$ by an infinite-dimensional normed vector space, the statement we obtain is false.)