MAA 4211, Fall 2020–Assignment 1's non-book problems

B1. Let A and B be sets and let $f : A \to B$ be a bijective function. Show that the definition of the inverse function $f^{-1} : B \to A$ given in class is equivalent to the one given in Bartle & Sherbert (Definition 1.1.11). I.e. show that if $g : Y \to X$ is a function, then g satisfies the criteria "$g \circ f = \mathrm{id}_X$ and $f \circ g = \mathrm{id}_Y$" if and only the graph of g is the set called g in Bartle & Sherbert's definition. (Recall that the *graph* of a function $h : X \to Y$ is the subset $\{(x, h(x)) \mid x \in X\} \subseteq X \times Y\}$.)

B2. Let X and Y be nonempty sets. Show that there exists an injective map $f : X \to Y$ if and only if there exists a surjective map $g : Y \to X$.

B3. Find a formula expressing the cardinality of the Cartesian product of two finite sets X, Y in terms of the cardinalities of X and Y. Justify your answer.

B4. Recall that the *power set* of a set X is the set $\mathcal{P}(X)$ of all subsets of X (an *element* of $\mathcal{P}(X)$ is a *subset* of X). For every set X, let 2^X denote the set of all functions from X to the two-element set $\{0, 1\}$.

(a) Let X be a set. Exhibit a bijection from $\mathcal{P}(X)$ to 2^X (or from 2^X to $\mathcal{P}(X)$). Thus $\mathcal{P}(X)$ has the same cardinality as 2^X.

(b) If X is a finite set of cardinality n, what is the cardinality of 2^X? (Derive your formula; don't just write down the answer.) If your answer is correct, you will see where the notation "2^X" comes from.

In view of part (a), the answer to part (b) also gives the cardinality of $\mathcal{P}(X)$ for any finite set.