MAA 4211, Fall 2020—Assignment 2's non-book problems

B1. Let $X := (x_n)_{n=1}^{\infty}$ be a sequence in $\mathbf{R}$, and define sequences $Y := (y_n)_{n=1}^{\infty}$ and $Z := (z_n)_{n=1}^{\infty}$ by

$$\begin{aligned} y_n &= x_{2n-1} \quad \text{for each } n \in \mathbf{N}, \\ z_n &= x_{2n} \quad\quad \text{for each } n \in \mathbf{N}. \end{aligned}$$

(In other words, Y and Z are the subsequences of X given by the odd-numbered terms and even-numbered terms, respectively.) Prove that the following are equivalent:

(i) X converges.

(ii) Both Y and Z converge, and their limits are equal.

Prove also that if condition (ii) holds, then $\lim(X) = \lim(Y) = \lim(Z)$.

B2. Let $X := (x_n)_{n=1}^{\infty}$ be the sequence defined recursively by

$$\begin{aligned} x_1 &= \frac{1}{2}\,, \\ x_{n+1} &= \frac{1}{2 + x_n} \quad \text{for each } n \in \mathbf{N}. \end{aligned}$$

(So X is the sequence

$$\frac{1}{2}\,,\ \frac{1}{2+\frac{1}{2}}\,,\ \frac{1}{2+\frac{1}{2+\frac{1}{2}}}\,,\ \frac{1}{2+\frac{1}{2+\frac{1}{2+\frac{1}{2}}}}\,,\ \dots\,,$$

an example of something call a *continued fraction*.) Prove that X converges and find its limit.

Hint: prove that the even-numbered subsequence and odd-numbered subsequence both converge and that their limits are the same. Then apply problem B1.

Warning: You cannot prove that a limit (or anything else) exists by assuming it exists. However, before you get to the proof stage, there's nothing wrong asking yourself, "*If* the limit existed, what would it have to be?" Intelligent guesswork is part of problem-solving. Just don't forget that even if assuming the limit exists leads to only one possible value for it, that fact doesn't prove that the limit exists.

B3. (a) Let Z be any nonempty set, and let $\text{Func}(Z, \mathbf{R})$ denote the set of all functions from Z to $\mathbf{R}$. As temporary notation, just for this problem, let $\underline{\mathbf{0}}$ denote the constant function with value 0 (i.e. $\underline{\mathbf{0}}(z) = 0$ for all $z \in Z$). For $f, g \in \text{Func}(Z, \mathbf{R})$ and $c \in \mathbf{R}$ we define elements $f + g \in \text{Func}(Z, \mathbf{R})$ and $cf \in \text{Func}(Z, \mathbf{R})$ by

$$\begin{aligned} f + g &= \text{the function "}z \mapsto f(z) + g(z)\text{"}, && (1) \\ cf &= \text{the function "}z \mapsto c \cdot f(z)\text{"}. && (2) \end{aligned}$$

Check that, with the operations above, $\text{Func}(Z, \mathbf{R})$ is a vector space with zero-element $\underline{\mathbf{0}}$. (Look up the definition of "vector space", which you probably haven't reviewed since you took MAS 4105, to make sure you're checking everything that needs to be checked.)

(b) Let $\mathbf{R}^\infty$ denote $\text{Func}(\mathbf{N}, \mathbf{R})$—i.e. the set of all real-valued sequences. The idea behind the notation "$\mathbf{R}^\infty$" is that can think of a sequence in $\mathbf{R}$ as an "ordered ∞-tuple", or infinite list, of real numbers. When we have this mental point of view, we often put a left-parenthesis in front of the list, and sometimes at the end, as in

$$(x_1\ ,\ x_2\ ,\ x_3\ , \dots$$

or

$$(x_1\ ,\ x_2\ ,\ x_3\ , \dots\ ).$$

By part (a), $\mathbf{R}^\infty$ is a vector space (when we equip $\mathbf{R}^\infty$ with the operations defined in equations (1) and (2)).

(i) Check that these operations correspond precisely to the notations "$X + Y$" and "cX" introduced in class (and in B&S, p.63). I.e. check that if $X = (x_n)_{n=1}^\infty$ and $Y = (y_n)_{n=1}^\infty$ are real-valued sequences, and $c \in \mathbf{R}$, then $X + Y$ as defined by equation (1) is the sequence whose n^{th} term is $x_n + y_n$, and that cX as defined by equation (2) is the sequence whose n^{th} term is cx_n.

(ii) In "list form", what is the zero element of $\mathbf{R}^\infty$?

(c) Let $\mathbf{R}_b^\infty \subseteq \mathbf{R}^\infty$ denote the set of *bounded* real-valued sequences. Show that $\mathbf{R}_b^\infty$ is a vector subspace of $\mathbf{R}^\infty$.

Remark. For $n \in \mathbf{N}$, writing elements of $\text{Func}(\mathbf{N}_n, \mathbf{R})$ (where $\mathbf{N}_n = \{1, 2, 3, \dots, n\}$) in "list form"—in this case, a finite list—we see that there is a natural bijection from $\text{Func}(\mathbf{N}_n, \mathbf{R})$ to $\mathbf{R}^n$. Under this bijection, the operations on $\text{Func}(\mathbf{N}_n, \mathbf{R})$ correspond to the usual vector-space operations on $\mathbf{R}^n$. This is additional motivation for the notation "$\mathbf{R}^\infty$".