**MAA 4211, Fall 2020—Assignment 3's non-book problems**

B1. Let $X := (x_n)_{n=1}^{\infty}$ be a bounded sequence in $\mathbf{R}$. Prove that $\lim X$ exists if and only if $\liminf X = \limsup X$ (Proposition 15.2 in the lecture notes).

B2. Let $X$ be a bounded sequence in $\mathbf{R}$.

(a) Prove that $X$ has a subsequence converging to $\limsup X$.

(b) Let $\alpha = \limsup X$. In view of part (a), $X$ has a subsequence converging to $\alpha$. Prove that $\alpha$ is the largest real number with this property. I.e. prove that if $c > \alpha$, then there exists no subsequence of $X$ converging to $c$.

(c) State the analog of parts (a) and (b) for "lim inf". Do not write out the analogous proofs, but summarize briefly what changes would be required in your proofs of (a) and (b) to prove these analogs.

B3. [This problem has been removed; it asked you to prove something that's false.]

B4. Let $X$ and $Y$ be bounded sequences in $\mathbf{R}$.

(a) Prove that

$$\limsup(X+Y) \ \leq \ \limsup X + \limsup Y.$$

(b) Give an example showing that the inequality in part (a) can be strict.

B5. Let $(x_n)_{n=1}^{\infty}$ be the sequence defined recursively by

$$\begin{aligned} x_1 &= 999, \\ x_{n+1} &= \frac{x_n}{2} + \frac{1}{x_n} + 1 \quad \text{for } n \in \mathbf{N}. \end{aligned}$$

(a) Prove that this sequence converges.

(b) Compute $\lim_{n\to\infty} x_n$ and prove that your answer is correct.(If you do this problem the way I'm expecting, the way you figure out the value of the limit should amount to a proof that your answer is correct. You should also find that the "999" could be replaced with any other positive number without affecting convergence or the value of the limit.)

B6. Find all cluster points of the subset $S$ of $\mathbf{R}$ defined by $S = \{\frac{1}{n} + \frac{1}{m} \mid n, m \in \mathbf{N}\}$, and prove that you have, indeed, found *all* the cluster points of $S$.

(You're going to find that the second part—proving that there are no cluster points other than the ones you've found—is enormously harder than proving that your found points are cluster points. Don't be surprised if you find it harder than anything else I've asked you to prove before.)

B7. (Do the reading in part C of this assignment first.) Prove Theorem 4.3.3 in Bartle & Sherbert.

B8. Let $I \subseteq \mathbf{R}$ be an interval and let $b \in I$. Let $I_- = \{x \in I : x \leq b\}$ and $I_+ = \{x \in I : x \geq b\}$. Let $f : I \to \mathbf{R}$ be a function. [Last sentence was omitted from original statement of problem.]

(a) Prove that if $f|_{I_+}$ and $f|_{I_-}$ are continuous, then $f$ is continuous.

(b) Prove that if $f|_{I_+}$ and $f|_{I_-}$ are uniformly continuous, then $f$ is uniformly continuous.

B9. Let $A \subset \mathbf{R}$. Given functions $f, g : A \to \mathbf{R}$, we define functions $\max\{f, g\} : A \to \mathbf{R}$ and $\min\{f, g\} : A \to \mathbf{R}$ by

$$\max\{f,g\}(x) = \max\{f(x), g(x)\} \quad \text{and} \min\{f,g\}(x) = \min\{f(x), g(x)\} \quad \text{for all } x \in A.$$

Show that if $f$ and $g$ are continuous, then so are $\max\{f, g\}$ and $\min\{f, g\}$. (*Hint*: First show that for all $a, b \in \mathbf{R}$, we have $\max\{a, b\} = \frac{1}{2}(a + b + |a - b|)$ and $\min\{a, b\} = \frac{1}{2}(a + b - |a - b|)$.)

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**Problems B10 and B11 are *recommended* rather than *required*.**

B10. Let $A \subseteq \mathbf{R}$, let $c$ be a cluster point of $A$, and let $f : A \to \mathbf{R}$. Call $f$ *locally bounded at $c$* if $f$ is bounded on some neighborhood of $c$ (i.e. if for some $\delta_1 > 0$, the restriction of $f$ to $V_{\delta_1}(c) \cap A$ is bounded).

(a) Assume that $f$ is locally bounded at $c$, and let $\delta_1 > 0$ be such that the restriction of $f$ to $V_{\delta_1}(c) \cap A$ is bounded. Define $w_f : (0, \delta_1) \to \mathbf{R}$ by

$$\begin{aligned} w_f(\delta) &:= \sup(\, f(V_\delta(c) \cap A)\,) - \inf(\, f(V_\delta(c) \cap A)\,) \\ &= \sup\left\{f(x) \mid x \in A \ \text{ and } \ |x - c| < \delta\right\} \ - \ \inf\{f(x) \mid x \in A \ \text{ and } \ |x - c| < \delta\}. \end{aligned}$$

Show that $w_f$ is an increasing function.

From a result proven in class (Proposition 25.1), it follows that $\lim_{\delta \to 0} w_f(\delta)$ exists.

(b) Assume that $f$ is locally bounded at $c$. Note that if if $\delta_1, \delta_2$ are any positive numbers for which the restriction of $f$ to $V_{\delta_i}(c) \cap A$ is bounded for each $i \in \{1, 2\}$, then the two functions $\tilde{w}_f : (0, \delta_i) \to \mathbf{R}$ define as in part (a) coincide on the interval $(0, \min\{\delta_1, \delta_1\})$, and hence have the same limit at 0.

We define the *oscillation of $f$ at $c$* to be

$$\mathrm{osc}_c(f) = \lim_{\delta \to 0} w_f(\delta) \ .$$

The preceding paragraph shows that $\mathrm{osc}_c(f)$ is *well-defined*: it depends only on the function $f$ and cluster point $c$, not on the auxiliary number $\delta_1$ used to define the domain of $w_f$.

Show that $\lim_c f$ exists if and only if $f$ is bounded on some neighborhood of $c$ and $\mathrm{osc}_c(f) = 0$.

(d) For the function $f : \mathbf{R}\backslash\{0\} \to \mathbf{R}$ defined by $f(x) = \sin(\frac{1}{x})$, compute $\mathrm{osc}_0(f)$. (You may assume that the sine function has its familiar properties.)

B11. Let $I \subseteq \mathbf{R}$ be an interval, let $f : I \to \mathbf{R}$ be a monotone function, and let $c \in I$. Show that $f$ is bounded on $V_\delta(c) \cap I$ for every $\delta > 0$, and that

$$\mathrm{osc}_c(f) = \pm j_f(c). \tag{1}$$

When do we get the plus sign in equation (1), and when do we get the minus sign?