MAA 4211, Fall 2020—Assignment 4's non-book problems

B1. Let $I \subseteq \mathbf{R}$ be an interval, let $x_0 \in I$, and $f : I \to \mathbf{R}$ be a function that is continuous on I and differentiable on $I \backslash \{x_0\}$.

Below, be careful not to assume that f has any additional properties. For example, don't assume that f' is continuous on $I \backslash \{x_0\}$.

Hints: (1) The Mean Value Theorem is a great theorem. (2) If you're unsure how the MVT might be relevant, it may be helpful to do part (b) of this problem before part (a).

(a) Assume that x_0 is an interior point of I, and that that $\lim_{x \to x_0^+} f'(x)$ and $\lim_{x \to x_0^-} f'(x)$ exist and are equal. Prove that f is differentiable at x_0 and that $f'(x_0)$ has the same value as these two limits (and hence that f' not only exists at x_0 but is continuous there).

(b) Assume x_0 is an endpoint of I. If x_0 is a left endpoint of I, assume that $\lim_{x \to x_0^+} f'(x)$ exists; if x_0 is a right endpoint of I, assume that $\lim_{x \to x_0^-} f'(x)$ exists. Prove that f is differentiable at x_0 and that $f'(x_0)$ has the same value as the corresponding limit above (and hence, again, that f' not only exists at x_0 but is continuous there).

B2. Let $I \subseteq \mathbf{R}$ be a positive-length (i.e. non-singleton) interval, and let $f : I \to \mathbf{R}$ be a function.

(a) Let $x_0 \in I$. We say that f is *Lipschitz at* x_0 if there exist $\delta > 0$ and $K \in \mathbf{R}$ such that for all $x \in V_\delta(x_0) \cap I$ we have $d(f(x), f(x_0)) \leq K d(x, x_0)$ (equivalently, $|f(x) - f(x_0)| \leq K|x - x_0|$).

Prove that if f is differentiable at x_0, then f is Lipschitz at x_0.

(b) Prove that if f is differentiable, and the function $f' : I \to \mathbf{R}$ is bounded, then f is Lipschitz.

(c) We call f *locally Lipschitz* if for all $x \in I$ there exists $\delta > 0$ such that the restriction of f to $V_\delta(x)$ is Lipschitz.

Prove that if f is continuously differentiable, then f is locally Lipschitz.

(*Note*: For a given $x_0 \in I$, the condition that "The restriction of f to $V_\delta(x_0) \cap I$ is Lipschitz" is stronger—i.e. more restrictive—than "f is Lipschitz at x_0." For $f|_{V_\delta(x_0) \cap I}$ to be Lipschitz, we need existence of some $K \in \mathbf{R}$ such that $\underline{\text{for all } x_1, x_2} \in V_\delta(x_0) \cap I$, we have $d(f(x_1), f(x_2)) \leq K d(x_1, x_2)$. For f to be Lipschitz at x_0, we only need there to be some $K_0 \in \mathbf{R}$ such that $\underline{\text{for all } x_1} \in V_\delta(x_0) \cap I$, we have $d(f(x_1), f(x_0)) \leq K_0 d(x_1, x_0)$. Even if f is Lipschitz at every point $x_0 \in I$, this K_0 could depend on x_0. For $f|_{V_\delta(x_0) \cap I}$ to be Lipschitz, there must be a K that's independent of x_0. Thus, "locally Lipschitz" is a stronger condition than "Lipschitz at every point".)

B3. Let $a, b \in \mathbf{R}$ and assume $a < b$.

(a) Assume that $f, g : [a, b) \to \mathbf{R}$ are continuous, and are differentiable on (a, b). Assume also that $f(a) = g(a)$ and that $f'(x) > g'(x)$ for all $x \in (a, b)$. Prove that $f(x) > g(x)$ for all $x \in (a, b)$.

(b) Assume that $f, g : (a, b] \to \mathbf{R}$ are continuous, and are differentiable on (a, b). Assume also that $f(b) = g(b)$ and that $f'(x) > g'(x)$ for all $x \in (a, b)$. In this case, what order-relation do $f(x)$ and $g(x)$ obey for $x \in (a, b)$?

In part (b), you are not being asked for a formal proof. Just state how, under the hypotheses in (b), any inequalities you used for the proof in part (a) become modified, and how this affects or does not affect the conclusion.

In problems B4 and B5(c), you may assume that the sine and cosine functions have the derivatives you learned in Calculus I, as well as their usual trigonometric properties.

B4. Prove that, for all $x > 0$,

$$
\begin{aligned}
&\text{(a)} \quad \sin x \; < \; x, \\
&\text{(b)} \quad \cos x \; > \; 1 - \frac{x^2}{2}\,, \qquad \text{and} \\
&\text{(c)} \quad x - \frac{x^3}{3!} \; < \; \sin x \; < \; x - \frac{x^3}{3!} + \frac{x^5}{5!}\,.
\end{aligned}
$$

Hint for B4(a): Problem B3(a). Just FYI: when your professor was a lad, problems like proving the inequality in B4(a) were standard problems on AP Calculus BC exams.

Hint for B4(b): B3(a) plus B4(a).

Hint for B4(c): Apply previous ideas several times.

Note: You may recognize the polynomials appearing above as Taylor polynomials (based at 0) of the sine or cosine function, of various orders. A very pretty fact is that the pattern you see in B3(c) for the sine function (and the analogous one for cosine that you should find yourself needing to prove) continues for the higher-degree Taylor polynomials: sine and cosine are "squeezed" between their successive Taylor polynomials. We have not yet defined Taylor polynomials, let alone proven any version of Taylor'sTheorem, so anything with "Taylor" in it is off-limits to you in this problem. But even if we *did* have Taylor's Theorem available to us now, using it wouldn't be the best way to do this problem.

B5. In class we proved that if $I \subseteq \mathbf{R}$ is an interval, $f : I \to \mathbf{R}$ is continuous on I and differentiable on I°, and $f'(x) > 0$ for all $x \in I^\circ$, then f is strictly increasing (i.e. $x_1 < x_2$ implies $f(x_1) < f(x_2)$). (Here I° is alternative notation for the interior of I, which I denoted $\overset{\circ}{I}$ in class. As you can see, putting the circle on top of the I in LaTeX doesn't yield great-looking results.) In this problem we show that the requirement "$f'(x) > 0$ for all $x \in I^\circ$" can be somewhat relaxed without affecting the conclusion. Parts (a) and (b) draw successively stronger conclusions by using successively weaker hypotheses. Each of the first two problem-parts is intended to help you do the next part.

(a) Let $a, b \in \mathbf{R}$, with $a < b$. Let I be any of the intervals $[a, b), (a, b]$, or $[a, b]$. Assume that $f : I \to \mathbf{R}$ is continuous, is differentiable on the open interval (a, b), that $f'(x) \geq 0$ for all $x \in (a, b)$, and that $f'(x) = 0$ for at most finitely many $x \in (a, b)$. Prove that f is strictly increasing on I.

(b) Let $I \subseteq \mathbf{R}$ be a nonempty positive-length interval (not necessarily bounded). Assume that $f : I \to \mathbf{R}$ is differentiable and that $f'(x) \geq 0$ for all $x \in I$. Let $Z(f') = \{x \in I \mid f'(x) = 0\}$ (the *zero-set* of f'), and assume that $Z(f')$ has no cluster points in in I. (Note that if I is not closed, we are not ruling out cluster points of I that don't lie in I, i.e. endpoints of I that don't lie in I.) Prove that f is strictly increasing on I.

(c) Define $f : \mathbf{R} \to \mathbf{R}$ by $f(x) = x - \sin x$. Prove that f is strictly increasing.

B6. Let $f : [a, b] \to \mathbf{R}$.

(a) Prove that if f is Riemann integrable, then for any sequence $(\dot{\mathcal{P}}_n)_{n=1}^\infty$ of tagged partitions of $[a, b]$ for which $\|\mathcal{P}_n\| \to 0$ as $n \to \infty$,

$$\lim_{n\to\infty} S(f; \dot{\mathcal{P}}_n) = \int_a^b f. \tag{1}$$

(Hence the integral can be evaluated by taking such a limit, *if you know ahead of time that f is integrable.*)

(b) Assume that for every sequence $(\dot{\mathcal{P}}_n)_{n=1}^\infty$ of tagged partitions of $[a, b]$ for which $\|\mathcal{P}_n\| \to 0$ as $n \to \infty$, $\lim_{n\to\infty} S(f; \dot{\mathcal{P}}_n)$ exists. Prove that f is Riemann integrable on $[a, b]$, and that for every such sequence $(\dot{\mathcal{P}}_n)$, the equality (1) holds.

(Thus, taken together, parts (a) and (b) say that (i) f is Riemann integrable if and only if for every sequence $(\dot{\mathcal{P}}_n)_{n=1}^\infty$ of tagged partitions for which $\|\mathcal{P}_n\| \to 0$ as $n \to \infty$, $\lim_{n\to\infty} S(f; \dot{\mathcal{P}}_n)$ exists and (ii) in the integrable case, the value of each such limit of Riemann sums is $\int_a^b f$.)