MAA 4211, Fall 2020—Assignment 5's non-book problems

B1. Let $c > 0$. Prove that $\lim_{n\to\infty} c^{1/n} = 1$.

Note: At the time this exercise is being assigned, we have not defined any logarithmic function (ln, $\log_2$, etc.) or exponential function (e.g. $x \mapsto e^x$ or $x \mapsto 2^x$, with domain larger than $\mathbf{Q}$), let alone derived any properties of such functions. So, in this exercise, you may not make use of any such function.

B2. Let X be a nonempty set, let $(f_n : X \to \mathbf{R})_{n=1}^{\infty}$ be a sequence of functions, and let $f : X \to \mathbf{R}$ be a function. Assume that there is a real-valued sequence $(c(n))_{n=1}^{\infty}$ such that (i) for all $n \in \mathbf{N}$ and $x \in X$, we have $d(f_n(x), f(x)) \leq c(n)$, and (ii) $\lim_{n\to\infty} c(n) = 0$. Prove that (f_n) converges uniformly to f.

Thus, to prove that a sequence (f_n) converges uniformly to a given function f, it suffices to find, for each n, a uniform upper bound $c(n)$ on the distances $d(f_n(x), f(x))$ (where "uniform" means "independent of x"), with the property that $c(n) \to 0$ as $n \to \infty$. In practice, this is virtually always how uniform convergence is shown (for a sequence of functions that *does* converge uniformly).