MAA 4211, Fall 2021–Assignment 1's non-book problems

In doing these (or any) proof-problems, make sure you do not make any assumptions that are not in the hypotheses! For example, if the hypotheses say "Let Z be a set" but don't say anything else about Z, don't assume that Z is a set of *numbers*, or a finite set, or a countably infinite set (one whose elements can be enumerated $\{z_1, z_2, z_3, \ldots\}$).

B1. Let X and I be sets, and let $\{A_i\}_{i\in I}$ be a family of subsets of X indexed by I. Prove the "generalized De Morgan Laws":

$$\left(\bigcup_{i\in I} A_i\right)^c = \bigcap_{i\in I} (A_i)^c$$

and

$$\left(\bigcap_{i\in I} A_i\right)^c = \bigcup_{i\in I} (A_i)^c$$

(The superscript "c" denotes complementation in X.)

B2. (Before doing this problem, do the reading in part C of this assignment, and do Abbott exercises 1.2/ 7,9. *Make sure you understand what this problem is about*—**images** *and* **inverse images** *of sets under functions. There is no such thing as "proof by notation".*)

(a) Determine, with proof, which (if any) of the following statements is/are true for all sets X, Y, functions $f : X \to Y$, and subsets $A \subseteq X$:

(i) $f^{-1}(f(A)) \subseteq A$; (ii) $f^{-1}(f(A)) \supseteq A$; (iii) $f^{-1}(f(A)) = A$.

(You're not being asked to prove that any false statement is false, just that any true statement is true.)

(b) In part (a), if statement (iii) is not always true, determine, with proof, the most general condition on the function f that makes (iii) true for all $A \subseteq X$. (In other words, find a condition such that (iii) is true if and only if f satisfies that condition.)

(c) Determine, with proof, which (if any) of the following statements is/are true for all sets X, Y, functions $f : X \to Y$, and subsets $B \subseteq Y$:

(i) $f(f^{-1}(B)) \subseteq B$; (ii) $f(f^{-1}(B)) \supseteq B$; (iii) $f(f^{-1}(B)) = B$.

(Again, you're not being asked to prove that any false statement is false, just that any true statement is true.)

(d) In part (c), if statement (iii) is not always true, determine, with proof, the most general condition on the function f that makes (iii) true for all $B \subseteq Y$.

B3. (Same instructions as at beginning of B2 apply.)

Let $f : X \to Y$ and $g : Y \to Z$ be functions, let $A \subseteq X$, and let $B \subseteq Z$.

(a) Prove that $(g \circ f)(A) = g(f(A))$.

(b) Prove that $(g \circ f)^{-1}(B) = f^{-1}(g^{-1}(B))$.