MAA 4211, Fall 2021–Assignment 2's non-book problems

- In the problems below:
  - For any sets A and B, the notation "$A \sim B$" means "there exists a bijection from A to B."
  - Earlier problems often provide results that will help with later problems.
- **Exception to the "no use of implication-arrows" rule** (on this and future assignments, as well as exams): If you're given an "if and only if" statement to prove, you may use the notation "($\Longrightarrow$)" (respectively, "($\Longleftarrow$)") to indicate that you're starting the proof of the forward (respectively, reverse) implication. Similarly, if you're asked to prove that several listed statements are equivalent, you may use notation such as "(ii) $\Longrightarrow$ (iii)" to indicate which implication you're about to start proving.

B1. Let X and Y be nonempty sets. Show that there exists an injective map $f : X \to Y$ if and only if there exists a surjective map $g : Y \to X$. ("Map" is another word for "function".)

> *Note*: In case you happen to know what the Axiom of Choice is: you're allowed to use it, and you don't have to state that you're using it.
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> If you *don't* know what the Axiom of Choice is, don't worry about what I just said. I guarantee you that you already believe the Axiom of Choice and that you'll assume it implicitly whenever it's needed.

B2. (a) Prove that the inverse of a bijection is a bijection. (Consequently, for any sets A and B, if $A \sim B$ then $B \sim A$.)

(b) Prove that the composition of two bijections is a bijection. (Consequently, for any sets A, B and C, if $A \sim B$ and $B \sim C$, then $A \sim C$.)

B3. Recall that, *by definition*, a nonempty set S is finite if $S \sim \{1, 2, \ldots, n\}$ for some $n \in \mathbf{N}$. In an early lecture, the following was asserted, with its proof left as an exercise (which is exactly this homework problem):

> For any set S, if $n, m \in \mathbf{N}$ are such that $S \sim \{1, 2, \ldots, n\}$ and $S \sim \{1, 2, \ldots, m\}$, then $m = n$.

Prove this assertion, **being careful to avoid circular reasoning.** What you're proving here is exactly what allowed us to *define* cardinality of a nonempty finite set. In your proof, neither the word "cardinality" nor any notation for it should appear, nor should any statements like "this set has n elements" or "this set has more [or fewer] elements than this other set," or "these sets have the same number of elements." (For finite sets, "number of elements" is a synonym for cardinality, so does not make sense until *after* the assertion above is proven.)

You *are* allowed to use the fact that if $n, m \in \mathbf{N}$, then $n < m$ if and only if $\{1, 2, \ldots, n\} \subsetneq \{1, 2, \ldots, m\}$.

B4. Prove that every subset of a finite set is finite.

B5. (Do Abbott exercise 1.5.1 first.) Prove the following proposition, which was stated in class on Friday, Sept. 24.

Proposition. *Let S be a set. Then the following are equivalent:*

(i) S is at most countable.

(ii) There exists an injective function from S to $\mathbf{N}$.

(iii) Either $S = \emptyset$ or there exists a surjective function from $\mathbf{N}$ to S.

(iv) Either $S = \emptyset$ or there exists a surjective function from some countably infinite set to S.

Note: (1) The combination of problem B4 above and Abbott exercise 1.5.1 proves the following fact: *Every subset of an at-most countable set is at most countable.* You may assume this fact when doing problem B5.

(2) : The implications "(iii) $\Longrightarrow$ (i)" and "(iv) $\Longrightarrow$ (i)" are very useful in giving efficient proofs that certain sets, or types of sets, are at most countable.

(3): *Often*, the most efficient way to prove the equivalence of several statements—say (i)–(iv), for the sake of concreteness—is to prove "(i) $\Longrightarrow$ (ii),", "(ii) $\Longrightarrow$ (iii)," "(iii) $\Longrightarrow$ (iv),", and "(iv) $\Longrightarrow$ (i)" (a strategy abbreviated as "(i) $\Longrightarrow$ (ii) $\Longrightarrow$ (iii) $\Longrightarrow$ (iv) $\Longrightarrow$ (i)"). With this strategy, there are only four one-way implications you need to prove, rather than 12 (one for each choice of the pair (A,B) in "A implies B"). But *sometimes* it's easier to proceed by proving some of the pairwise-equivalences directly, e.g. by proving "(i) $\Longleftrightarrow$ (ii)," "(ii) $\Longleftrightarrow$ (iii)," and "(iii) $\Longleftrightarrow$ (iv)." In particular, the latter strategy (or a hybrid strategy) may be more efficient if some of these "iff"s follow immediately from some previously proven result.

(4): Recall that for any set B, there is a (unique) function $f : \emptyset \to B$ (the "empty function with codomain B"), and that this function is injective. These facts become clear only when we use the set-theoretic definition of "function from A to set B": a subset $G \subseteq A \times B$ with the property that for each $a \in A$, there exists a unique $b \in B$ such that $(a, b) \in G$. When $A = \emptyset$, the Cartesian product $A \times B$ is also empty (there *are* no ordered pairs (a, b) with $a \in A$ and $b \in B$)), and the subset $\emptyset \subset \emptyset \times B = \emptyset$ satisfies the definition of "function from $\emptyset \to B$ *vacuously* (*every* statement of the form "For each $a \in A$, $\langle$blah blah blah$\rangle$" is true, because there *are* no elements $a \in \emptyset$). The condition for injectivity—that there do not exist distinct elements $a_1, a_2 \in \emptyset$ and an element $b \in B$ such that (a_1, b) and (a_2, b) both lie in $G = \emptyset$—is satisfied vacuously, since there are no elements $a_1, a_2 \in \emptyset$ *period*.

The situation is quite different for empty *co*domains. If $B = \emptyset$ but A is *non*empty, there are *no* functions from A to B, not even "empty functions". The only subset G of $A \times \emptyset = \emptyset$ is $\emptyset$ itself, and this subset does *not* have the property that for every $a \in A$, there is some $b \in B = \emptyset$ for which $(a, b) \in G = \emptyset$.

However, the function I called "$\mathrm{id}_\emptyset$" in class—the empty function with empty codomain—exists, and in addition to being injective vacuously, is surjective vacuously ("For each $b \in \emptyset$, $\langle$blah blah blah$\rangle$" is true no matter what "blah blah blah" is, because there *are* no elements $b \in \emptyset$). Thus we may (accurately) refer to this function as the "empty bijection".

B6. (a) Let A, B be sets and assume that $f : \mathbf{N} \to A$ and $g : \mathbf{N} \to B$ are surjective. Define $h : \mathbf{N} \to A \cup B$ by

$$h(n) = \begin{cases} f(\frac{n+1}{2}) & \text{if } n \text{ is odd,} \\ \\ g(\frac{n}{2}) & \text{if } n \text{ is even.} \end{cases}$$

(Thus $h(1) = f(1), h(2) = g(1), h(3) = f(2), h(4) = g(2)$, etc.) Show that h is surjective.

(b) Prove that if sets A and B are at most countable, then so is $A \cup B$.

(c) Prove that, for each $n \in \mathbf{N}$, if $A_1, \ldots, A_n$ are at-most countable sets, then $A_1 \cup A_2 \cup \cdots \cup A_n$ is at most countable.

Note: Part (b) is, of course, a step in the proof of (c). The result proven in (c) is a stronger version of Abbott's Theorem 1.5.8(i), with "countably infinite" [his "countable"] generalized to "at most countable". You're not allowed to assume any part of Abbott's Theorem 1.5.8; that theorem is a *special case* of what I am having you prove, and the proofs I'm leading you to, in this problem and the next, yield *more* than Theorem 1.5.8 with *less* work.

B7. *Fact you may assume for this problem*: $\mathbf{N} \times \mathbf{N}$ is countably infinite. (A proof was sketched in class.)

(a) Let $\{A_n\}_{n \in \mathbf{N}}$ be an infinite collection of nonempty at-most countable sets, indexed by $\mathbf{N}$. Prove that $\bigcup_{n=1}^{\infty} A_n$ is at most countable. (This is a stronger version of Abbott's Theorem 1.5.8(ii).)

Hint: Applying problem B5, you should be able to use the hypothesis that each of the sets A_n is nonempty and at-most countable to produce a surjective map $\mathbf{N} \times \mathbf{N} \to \bigcup_{n=1}^{\infty} A_n$.

(b) Let I be a nonempty, at-most countable set, and let $\{B_i\}_{i \in I}$ be a family of at-most countable sets, indexed by I. Prove that $\bigcup_{i \in I} B_i$ is at most countable.

Note: The conclusions of (a) and (b) are each often stated with the simple and memorable wording,

$$\text{The countable union of countable sets is countable.} \tag{1}$$

Statement (1) is a true statement with either of the standard conventions for what "countable" means. However, with the convention that "countable" means "countably infinite" (Abbott's convention), statement (1) is weaker than it is with the convention that "countable" means "finite or countably infinite" (for which I'm using the term "at most countable"). However, if "at most countable" is substituted for "countable" in statement (1), the statement becomes less pithy, and loses its punch.