MAA 4211, Fall 2021–Assignment 3's non-book problems

B1. A sequence $(a_n)_{n=1}^{\infty}$ in a set X is called *eventually constant* if, for some $N \in \mathbf{N}$, all terms with $n \geq N$ are equal (hence equal to a_N); equivalently, if there exist $N \in \mathbf{N}$ and $c \in \mathbf{R}$ (the *eventual value* of the sequence) such that for every $n \geq N$ we have $a_n = c$. Two sequences $(a_n)_{n=1}^{\infty}$, $(b_n)_{n=1}^{\infty}$ are is called *eventually equal* if, for some $N \in \mathbf{N}$, we have $a_n = b_n$ for each $n \geq N$. (Thus, another definition of "eventually constant" is "eventuallly equal to some constant sequence.")

(a) Let $A := (a_n)_{n=1}^{\infty}$ be an eventually constant sequence in $\mathbf{R}$. Show that A converges to its eventual value.

(b) Let $A := (a_n)_{n=1}^{\infty}$, $B := (b_n)_{n=1}^{\infty}$ be eventually equal sequences in $\mathbf{R}$. Show that either both sequences converge, or both diverge. In th convergent case, show that $\lim(A) = \lim(B)$.

B2. Let $(a_n)_{n=1}^{\infty}$ be a sequence in $\mathbf{R}$, let $L \in \mathbf{R}$, and consider the statement "$\lim_{n\to\infty} a_N = L$." Recall that the definition of this statement is: for all $\epsilon > 0$, there exists $N \in \mathbf{N}$ such that for all $n \geq N$, the inequality $|a_n - L| < \epsilon$ holds. There are a few other valid ways to word this definition.[1] But there are also a *lot* of invalid ways of wording it. Below are six of them. Three of them define *some* sequence-property (potentially a property that we've already defined some other, simpler way, and potentially a property that we haven't yet named). Two are gibberish that should make your head hurt, and to which you should be unable to attach a clear meaning. One is not quite gibberish—there actually *is* an unambiguous meaning—but it's not easy to figure out, and it's definitely *not* "$\lim_{n\to\infty} a_N = L$."

Determine, in each case, whether the given statement defines *some* sequence-property, or is simply gibberish. In the non-gibberish case(s), state more simply the property that the statement defines.

(a) There exists $N \in \mathbf{N}$ such that for all $\epsilon > 0$ and all $n \geq N$, we have $|a_n - L| < \epsilon$.

(b) We have $|a_n - L| < \epsilon$ for all $n \geq N$, for some $N \in \mathbf{N}$, for all $\epsilon > 0$.

(c) For some $N \in \mathbf{N}$, we have $|a_n - L| < \epsilon$ for all $n \geq N$, for all $\epsilon > 0$.

(d) For some $\epsilon > 0$, and all $N \in \mathbf{N}$, and all $n \geq N$, we have $|a_n - L| < \epsilon$.

(e) For $\epsilon > 0$ and $n \geq N$, we have $|a_n - L| < \epsilon$, where $N \in \mathbf{N}$.

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[1] For example, "for all $n \geq N$, the inequality $|a_n - L| < \epsilon$ holds" can be replaced by "$n \geq N$ implies $|a_n - L| < \epsilon$." Until you become experts at being able use universal quantifiers ("for all", "for every", etc.) and existential quantifiers ("for some", "there exists", etc.) correctly—never putting them in a wrong order, or locating them in the wrong part of a sentence, or omitting necessary quantifiers, etc.—I recommend sticking *only* to universal and existential quantifiers, and not using an "implies" phrase to replace a quantification.

(f) We have $|a_n - L| < \epsilon$, for $\epsilon > 0$, $n \geq N$, and $N \in \mathbf{N}$.

B3. A *reordering* of a sequence $(a_n)_{n=1}^{\infty}$ is a sequence of the form $(a_{f(n)})_{n=1}^{\infty}$, where $f : \mathbf{N} \to \mathbf{N}$ is a bijection. Said another way, a sequence $B := (b_n)_{n=1}^{\infty}$ is a reordering of $(a_n)_{n=1}^{\infty}$ if and only if there exists a bijection $f : \mathbf{N} \to \mathbf{N}$ such that for all $n \in \mathbf{N}$, the relation $b_n = a_{f(n)}$ holds.

Prove that if $A := (a_n)_{n=1}^{\infty}$ is a convergent sequence in $\mathbf{R}$, then every reordering of this sequence converges to the same value, namely $\lim_{n\to\infty} a_n$.

B4. Recall from class that a definition, more general than the one we've been using, of an (infinite) sequence in a set X, is a function from a domain of the form $\{n \in \mathbf{Z} : n \geq n_0\}$ to X. Such as sequence would usually written with notation such as $(a_n)_{n=n_0}^{\infty}$. The definitions of *convergence* and *limit* of such sequences are identical to the definitions for $n_0 = 1$. In this context, when we say that something about a_n is true for all $n \geq N$, it is implicit that we are taking $N \geq n_0$, since otherwise a_n would not be defined.

Given a sequence $A := (a_n)_{n=1}^{\infty}$, and any $n_0 \in \mathbf{N}$, we can consider the sequence $(a_n)_{n=n_0}^{\infty}$, often called a *tail* of A. (Each choice of n_0 determines a tail.)

(a) Prove that, for any real-valued sequence $A := (a_n)_{n=1}^{\infty}$, the following are equivalent:

(i) A converges.

(ii) At least one tail of A converges.

(iii) Every tail of A converges.

Also prove that, in the convergent case, every tail of A converges to $\lim(A)$. (This should come out in the wash while you're showing the equivalences.)

(b) Given a sequence $A := (a_n)_{n=1}^{\infty}$, and any $k \in \mathbf{N}$, we can consider the sequence $(a_{n+k})_{n=1}^{\infty}$. Such a sequence is also called a *tail* of A—the terms are $a_{k+1}, a_{k+2}, a_{k+3}, \ldots$—but is indexed differently from the tails defined above (it's still a function from $\mathbf{N}$ to $\mathbf{R}$, not a function from $\{n \in \mathbf{N} : n \geq k+1\}$ to $\mathbf{R}$).

For the sake of concreteness in this problem, call such tails "alternatively defined tails". Prove everything in part (a), with "tail" replaced by "alternatively defined tail".

B5. (a) Let Z be any nonempty set, and let $\text{Func}(Z, \mathbf{R})$ denote the set of all functions from Z to $\mathbf{R}$. As temporary notation, just for this problem, let $\underline{\mathbf{0}}$ denote the constant function with value 0 (i.e. $\underline{\mathbf{0}}(z) = 0$ for all $z \in Z$). For $f, g \in \text{Func}(Z, \mathbf{R})$ and $c \in \mathbf{R}$ we define elements $f + g \in \text{Func}(Z, \mathbf{R})$ and $cf \in \text{Func}(Z, \mathbf{R})$ by

$$\begin{aligned} f + g &= \text{the function "} z \mapsto f(z) + g(z) \text{"}, && (1) \\ cf &= \text{the function "} z \mapsto c \cdot f(z) \text{"}. && (2) \end{aligned}$$

Check that, with the operations above, $\text{Func}(Z, \mathbf{R})$ is a vector space with zero-element $\underline{\mathbf{0}}$. (Look up the definition of "vector space", which you probably haven't reviewed since you took MAS 4105, to make sure you're checking everything that needs to be checked.)

(b) Let $\mathbf{R}^\infty$ denote $\text{Func}(\mathbf{N}, \mathbf{R})$—i.e. the set of all real-valued sequences. The idea behind the notation "$\mathbf{R}^\infty$" is that we oftern think of a sequence in $\mathbf{R}$, informally, as an "ordered ∞-tuple", or infinite list, of real numbers. When we have this mental point of view, we often put a left-parenthesis in front of the list, and sometimes at the end, as in

$$(a_1\ ,\ a_2\ ,\ a_3\ , \dots$$

or

$$(a_1\ ,\ a_2\ ,\ a_3\ , \dots\ ).$$

By part (a), $\mathbf{R}^\infty$ is a vector space (when we equip $\mathbf{R}^\infty$ with the operations defined in equations (1) and (2)).

(i) Given $A = (a_n)_{n=1}^\infty, B = (b_n)_{n=1}^\infty \in \mathbf{R}^\infty$ and $c \in \mathbf{R}$, define sequences $A + B$ and cA by

$$A + B \ := (a_n + b_n)_{n=1}^\infty$$

and

$$cA := (ca_n)_{n=1}^\infty\ .$$

Check that these operations correspond precisely to the operations (1) and (2) when $Z = \mathbf{N}$. the notation "$A + B$" used in class. and "cX" introduced in class (and in B&S, p.63).

(ii) In "list form", what is the zero element of $\mathbf{R}^\infty$?

(c) Let $\mathbf{R}^\infty_b \subseteq \mathbf{R}^\infty$ denote the set of *bounded* real-valued sequences. Show that $\mathbf{R}^\infty_b$ is a (vector) subspace of $\mathbf{R}^\infty$. (Recall that given a vector space V, a subset $W \subseteq V$ is a subspace if and only if (i) the zero element of V lies in W, (ii) V is closed under addition, and (iii) V is closed under multiplication by scalars. The pair "(i) +(iii)" can be replaced by "(i)$'$+(iii"), where (i)$'$ is the statement that W is nonempty.)

(d) $\mathbf{R}^\infty_{\text{cvgt}} \subseteq \mathbf{R}^\infty$ denote the set of *convergent* real-valued sequences. Show that $\mathbf{R}^\infty_{\text{cvgt}}$ is a (vector) subspace of $\mathbf{R}^\infty$.

(e) Let $s : \mathbf{R}^\infty_b \to \mathbf{R}$ and $i : \mathbf{R}^\infty_b \to \mathbf{R}$ be the maps defined by $s(A) = \sup(\text{range}(A)) = \sup\{a_n : n \in \mathbf{N}\}$ (where $A = (a_n)_{n=1}^\infty$) and $i(A) = \inf(\text{range}(A))$. Let $\mathcal{L} : \mathbf{R}^\infty_{\text{cvgt}} \to \mathbf{R}$ be the map defined by $\mathcal{L}(A) = \lim A$. In linear-algebraic terms, what special type of map are s, i, and $\mathcal{L}$? What have we proven that shows this?

Remark. For $n \in \mathbf{N}$, writing elements of the set $\text{Func}(\mathbf{N}_n, \mathbf{R})$, where $\mathbf{N}_n = \{1, 2, 3, \dots, n\}$, in "list form"—in this case, a finite list—we see that there is a natural bijection from $\text{Func}(\mathbf{N}_n, \mathbf{R})$ to $\mathbf{R}^n$. Under this bijection, the operations on $\text{Func}(\mathbf{N}_n, \mathbf{R})$ correspond to the usual vector-space operations on $\mathbf{R}^n$. This is additional motivation for the notation "$\mathbf{R}^\infty$".

B6. (a) Let $a \in \mathbf{R}$, with $a > 1$. Show that the sequence $(a^n)_{n=1}^\infty$ is unbounded above.

Note: (1) **No infinite limits** (e.g. "$\lim_{n\to\infty} a^n = \infty$") should enter your proof. We have not defined what this limit-statment means yet. (2) **No logarithmic functions** should enter your argument, either explicitly or implicitly. We are a long way from being able to define logarithmic functions. (3) As always, **no circular reasoning**, or any argument that depends on something we haven't proven, is allowed. For example, something like "$\{a^n : n \in \mathbf{N}\}$ is unbounded above because a^n can be arbitrarily large" is circular reasoning; it simply restates the desired conclusion in another (still unproven) way.

(b) Let $b \in \mathbf{R}$ satisfy $|b| < 1$. Use part (a) to show that $\lim_{n\to\infty} b^n = 0$.

B7. In Abbott exercise 2.3.1(b) you are asked to prove that if $(x_n)_{n=1}^\infty$ is a sequence in $[0, \infty) \subseteq \mathbf{R}$ converging to x, then $x \geq 0$ and $\lim_{n\to\infty} \sqrt{x_n} = \sqrt{x}$. Find the mistake in the followng "proof" of this result.

" Since limits 'behave well' with respect to arithmetic,

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} \sqrt{x_n}\,\sqrt{x_n} = (\lim_{n\to\infty} \sqrt{x_n})(\lim_{n\to\infty} \sqrt{x_n}) = (\lim_{n\to\infty} \sqrt{x_n})^2$$

Hence $\lim_{n\to\infty} \sqrt{x_n} = \pm\sqrt{\lim_{n\to\infty} x_n}$.

Since $\sqrt{x_n} \geq 0$, we cannot have $\lim_{n\to\infty} \sqrt{x_n} < 0$. Thus if $\lim_{n\to\infty} x_n > 0$, we cannot have $\lim_{n\to\infty} \sqrt{x_n} = -\sqrt{\lim_{n\to\infty} x_n}$, so we must have $\lim_{n\to\infty} \sqrt{x_n} = \sqrt{\lim_{n\to\infty} x_n}$. If $\lim_{n\to\infty} x_n = 0$, then $\sqrt{\lim_{n\to\infty} x_n} = 0 = -\sqrt{\lim_{n\to\infty} x_n}$, so $\lim_{n\to\infty} \sqrt{x_n} = 0$, and again $\lim_{n\to\infty} \sqrt{x_n} = \sqrt{\lim_{n\to\infty} x_n}$. "

B8. (*Do Abbott exercises 2.3.5 and 2.4.2 first.*)
Let $X := (x_n)_{n=1}^\infty$ be the sequence defined recursively by

$$x_1 = \frac{1}{2}\,,$$
$$x_{n+1} = \frac{1}{2 + x_n} \quad \text{for each } n \in \mathbf{N}.$$

(So X is the sequence

$$\frac{1}{2}\,,\ \frac{1}{2+\frac{1}{2}}\,,\ \frac{1}{2+\frac{1}{2+\frac{1}{2}}}\,,\ \frac{1}{2+\frac{1}{2+\frac{1}{2+\frac{1}{2}}}}\,,\ \dots\,,$$

an example of something call a *continued fraction*.) Prove that X converges and find its limit.

Hint: prove that the even-numbered subsequence and odd-numbered subsequence both converge and that their limits are the same. Then apply Abbott exercise 2.3.5.

Warning: You cannot prove that a limit (or anything else) exists by assuming it exists. However, before you get to the proof stage, there's nothing wrong with asking yourself, "*If* the limit existed, what would it have to be?" Intelligent guesswork is part of problem-solving. Just don't forget that even if assuming the limit exists leads to only one *possible* value for it, that fact doesn't prove that the limit exists. (See, for example, Abbott exercise 2.4.2(a).)