**MAA 4211, Fall 2021–Assignment 4's non-book problems**

Throughout, "series" always means "series of real numbers".

B1. (a) Let $A := (A_n)_{n=1}^{\infty}$ be a sequence in $\mathbf{R}$, and assume $A$ is not eventually constant. Let $B$ be the subsequence of $A$ formed by removing "stutters": let $b_1 := a_1$; let $b_2$ be the first term of $A$ not equal to $b_1$; let $b_3$ be the next term of $A$ not equal to $b_2$; etc. More explicitly, $B := (a_{n_i})_{i=1}^{\infty}$, where $(n_i)_{i=1}^{\infty}$ is the strictly increasing sequence in $\mathbf{N}$ defined recursively as follows:

1. Let $n_1 = 1$.
2. For a given $i \in \mathbf{N}$, if $n_1, \ldots, n_i$ have been defined, let

$$n_{i+1} = \min\{n \in \mathbf{N} : n > n_i \text{ and } a_n \neq a_{n_i}\}. \tag{1}$$

   (The set on the right-hand side of (1) is nonempty since, otherwise, the sequence $A$ would be eventually constant. The well-ordering principle for $\mathbf{N}$ therefore assures us that this subset of $\mathbf{N}$ has a minimal element.)

Prove that $A$ converges if and only if $B$ converges, and that in the convergent case, $\lim A = \lim B$.

(b) Let $C := (c_n)_{n=1}^{\infty}$ be a sequence in $\mathbf{R}$, and let $S = \{n \in \mathbf{N} : c_n \neq 0\}$. Define the notation "$\sum_{n \in S} c_n$" by

$$\sum_{n \in S} c_n = \begin{cases} \sum_{i=1}^{\infty} c_{n_i} & \text{if } S \text{ is infinite and } (n_i)_{i=1}^{\infty} \text{ is the strictly increasing} \\ & \quad \text{sequence in } \mathbf{N} \text{ with range } S; \\ \sum_{i=1}^{k} c_{n_i} & \text{if } S \text{ finite and nonempty, and } n_1 < n_2 < \cdots < n_k \\ & \quad \text{are the elements of } S; \\ 0 & \text{if } S \text{ is empty.} \end{cases}$$

(i) Show that if $S$ is infinite, then $\sum_{n=1}^{\infty} c_n$ converges if and only if $\sum_{n \in S}$ converges. Show also that, in the convergent case, $\sum_{n=1}^{\infty} c_n = \sum_{n \in S} c_n$ .

*Note*: There is a reason that part (a) was assigned before part (b)(i).

(ii) Show that if $S$ is finite, then $\sum_{n=1}^{\infty} c_n$ converges and that, again, $\sum_{n=1}^{\infty} c_n = \sum_{n \in S} c_n$ .

**Remark.** Problem B1(b) formalizes the principle that "in infinite series, terms that are zero don't make a difference"—i.e. they affect neither *whether* a series converges, nor, in the convergent case, the *value of the sum*.

B2. (Extending Abbott's Theorem 2.7.7.) A series $\sum_{n=n_0}^{\infty} a_n$ (where $n_0 \in \mathbf{Z}$) is called an *alternating series* if the terms are all nonzero and alternate *strictly* in sign (the signs of $a_n$ and $a_{n+1}$ are opposite for all $n \geq n_0$). A common example is the *alternating harmonic series* $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \ldots$.

Abbott's Theorem 2.7.7 (in which $n_0 = 1$) implicitly broadens the definition of "alternating series" to include some series whose terms are eventually zero. For the sake of the theorem below, we will allow the same broader meaning for "alternating series". Specifically, given an eventually-zero sequence $(a_n)_{n=n_0}^{\infty}$, we will still call $\sum_{n=n_0}^{\infty} a_n$ an "alternating series" if, for some natural number $N \geq n_0$, (i) $a_n = 0$ for all $n \geq N$, but (ii) the terms $a_n$ with $n < N$ are all nonzero and alternate strictly in sign.

**"Alternating Series Theorem".**[1] Let $\sum_{n=1}^{\infty} a_n$ be an alternating series for which $|a_1| \geq |a_2| \geq |a_3| \geq \ldots$ (i.e. $|a_n| \geq |a_{n+1}|$ for all $n \geq 1$) and for which $\lim_{n\to\infty} a_n = 0$. Then $\sum_{n=1}^{\infty} a_n$ converges to a value that is (non-strictly) between 0 and $a_1$:

$$\text{if } \; a_1 > 0, \;\text{ then } \quad 0 \;\leq\; \sum_{n=1}^{\infty} a_n \;\leq\; a_1 \; ; \tag{2}$$

$$\text{if } \; a_1 < 0, \;\text{ then } \quad a_1 \;\leq\; \sum_{n=1}^{\infty} a_n \;\leq\; 0. \tag{3}$$

(a) Prove the "Alternating Series Theorem" by considering the even-indexed and odd-indexed subsequences of the sequence $(s_n)_{n=1}^{\infty}$ of partial sums of $\sum_{n=1}^{\infty} a_n$, and using the result of Abbott exercise 2.3.5 (part of the previous homework assignment). *Hint*: show that each of these partial-sum subsequences is monotone and bounded. Towards this end, group the terms into pairs of consecutive terms, possibly with unpartnered terms at the beginning or end (a necessity if $n$ is odd). Once you've shown that $\lim_{n\to\infty} s_{2n}$ and $\lim_{n\to\infty} s_{2n+1}$ both exist, you can check whether they're equal by examining the difference of these limits.

(b) Show that this theorem implies an analogous theorem for alternating series whose initial index is an arbitrary $n_0 \in \mathbf{Z}$. (Thus, there is effectively no loss in generality by our having stated the "Alternating Series Theorem" just for series with initial index 1.)

**Discussion before part (c).** The importance of statements (2) and (3) comes from their application to *estimating* sums of series that satisfy the hypotheses of the theorem.

[1] Abbott's Theorem 2.7.7 deals only with the case $a_1 \geq 0$ (a consequence of his hypotheses (i) and (ii)), and asserts only that when the series "passes the test"—i.e. when the hypotheses are satisfied— the series converges; Theorem 2.7.7 does not include any conclusions like statements (2) or (3) in the theorem stated above. Since these statements have nothing to do with *testing* for convergence, I've chosen the nickname "Alternating Series *Theorem*" for this result instead. Neither Abbott's nickname nor mine is standard.

For *any* convergent series $\sum_n a_n$, if we want to know the value of the series but we don't have an exact formula for it, the best we can do is to estimate this value by adding up the first $N$ terms for some $N$—i.e., by taking the $N^{\text{th}}$ partial sum $s_N$ as an estimate of the value of the full series. Of course, an estimate is not of much use if we have no way of telling how close it is to the value we're trying to estimate. We define the *error* of an estimate of any quantity by

$$\text{error} = (\text{true value of quantity}) - (\text{estimate}).$$

The absolute value of the error is a measure of how good the estimate is; the sign of the error indicates whether our estimate is an underestimate or an overestimate.

Given a sequence $A := (a_n)_{n=1}^{\infty}$ for which the series $\sum_{n=1}^{\infty} a_n$ converges, if $s_N(A)$ denotes the $N^{\text{th}}$ partial sum and $E_N(A)$ denotes the corresponding error , then

$$E_N(A) = \sum_{n=1}^{\infty} a_n - s_N(A) = \sum_{n=1}^{\infty} a_n - \sum_{n=N}^{\infty} a_n = \sum_{n=N+1}^{\infty} a_n \ ,$$

the tail of the *series* corresponding to the tail $(a_n)_{n=N+1}^{\infty}$ of the *sequence* $A$.

**Part (c).** Observe that if the series $\sum_{n=1}^{\infty} a_n$ satisfies the hypotheses of the "Alternating Series Theorem", any tail of a series satisfies these hypotheses as well. (i) Using this fact and the theorem, what are the strongest conclusions you can draw about the absolute value and sign of the error $E_N(A)$ (where $N \in \mathbf{N}$ is arbitrary)? (ii) For the series $\sum_{n=1}^{\infty}(-1)^{n-1}/n^3$, how many terms would we add in order to estimate the value of the series to within $10^{-6}$? (iii) For the convergent "$p$-series" $\sum_{n=1}^{\infty} 1/n^3$, explain why adding up the same number of terms of as you used in (ii) would lead to an error for whose absolute value would be greater than the corresponding error for the alternating series in (ii).

B3 (inserted 11/8/21 to help with next problem). Let $\sum_{n=1}^{\infty} a_n$ be a conditionally convergent series.

(a) Show that the sequence $(a_n)_{n=1}^{\infty}$ has infinitely many positive and infinitely negative terms. I.e. show that each of the sets $S_+ := \{n \in \mathbf{N} : a_n > 0\}$ and $S_- := \{n \in \mathbf{N} : a_n > 0\}$ is infinite.

(b) Show that both $\sum_{n\in S_+} a_n$ and $\sum_{n\in S_-} a_n$ diverge, with their partial sums approaching $\infty$ and $-\infty$, respectively. (Notation for these two series is as defined in problem B1(b).)

B4 (renumbered 11/8/21). **Rearrangements of conditionally convergent series.** (You may assume the results of B3 to do this problem. Even with that hint, you are likely to find this problem difficult.)

Let $\sum_{n=1}^{\infty} a_n$ be a conditionally convergent series.

(a) Let $r \in \mathbf{R}$. Show that $\sum_{n=1}^{\infty} a_n$ has a rearrangement that converges to $r$.

(b) Show that $\sum_{n=1}^{\infty} a_n$ has a rearrangement $\sum_{n=1}^{\infty} b_n$ for which $\sum_{n=1}^{\infty} b_n = \infty$.

(c) Show that $\sum_{n=1}^{\infty} a_n$ has a rearrangement $\sum_{n=1}^{\infty} b_n$ for which $\sum_{n=1}^{\infty} b_n = -\infty$.

B5 (renumbered 11/8/21). **(Ratio Test).** The Ratio Test you learned in Calculus 2 says this:

*Let $(a_n)_{n=1}^{\infty}$ be a sequence for which $a_n \neq 0$ for all $n$.*

*(i) If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| < 1$, then $\sum_{n=1}^{\infty} a_n$ converges absolutely.*

*(ii) If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| > 1$, then $\sum_{n=1}^{\infty} a_n$ diverges.*

(Recall that a hypothesis of the form "$\lim_{n\to\infty} b_n <$ (whatever)," [or a similar statement with "$<$" replaced by "$>$", "$\leq$", or "$\geq$", there is an implicit "**Assume that** $\lim_{n\to\infty} b_n$ ***exists*** **and satisfies ... .**" In particular, in the Ratio Test above, we are assuming that $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$ exists.)

(a) In Abbott exercise 2.7.9, you proved statement (i) of the Ratio Test. Now prove statement (ii).

**Discussion to set up part (b).** Because of its reliance on a limit that may or may not exist, the Ratio Test is more restricted in scope than (it turns out) it needs to be. For example, suppose $\left(\frac{a_{n+1}}{a_n}\right)_{n=1}^{\infty}$ is the sequence $(\frac{1}{2}, \frac{1}{3}, \frac{1}{2}, \frac{1}{3}, \frac{1}{2}, \frac{1}{3}, \dots)$. Then $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$ does not exist, so the Ratio Test (as written above) does not apply. But $\left|\frac{a_{n+1}}{a_n}\right| \leq \frac{1}{2} < 1$ for all $n$, and if you look back at your proof of part (i) of the Ratio Test, you should see that the same argument still works.

Below, you will prove a stronger version of the Ratio Test that can handle examples such as the one above, using the fact that lim sup and lim inf exist for any bounded sequence in $\mathbf{R}$. Note that any sequence of nonnegative real numbers is bounded below, so for such a sequence, "bounded above" implies "bounded".

(b) (**"Enhanced Ratio Test"**.) Let $\sum_{n=1}^{\infty} a_n$ be a series in which $a_n \neq 0$ for all $n$.

(i)′ Assume that the sequence $\left(\left|\frac{a_{n+1}}{a_n}\right|\right)_{n=1}^{\infty}$ is bounded above and that $\limsup_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| < 1$. Prove that $\sum_{n=1}^{\infty} a_n$ converges absolutely.

(ii)$'$ Assume that for all $n$ sufficiently large, $|a_{n+1}| \geq |a_n| > 0$. Prove that $\sum_{n=1}^{\infty} a_n$ diverges.

(Remember: "Statement $P(n)$ is true for all $n$ sufficiently large" means "There exists $N \in \mathbf{N}$ such that $P(n)$ is true for all $n \geq N$.")

(c) Prove that statement (i)$'$ of the "Enhanced Ratio Test" implies statement (i) of the original Ratio Test. Also, give an example of a series that satisfies the hypotheses of (i)$'$ but not (i). (Thus, statement (i)$'$ is stronger than statement (i), since there are series that satisfy the hypotheses of (i)$'$ but not (i).)

(d) Prove that statement (ii)$'$ of the "Enhanced Ratio Test" implies statement (ii) of the original Ratio Test. Also, give an example of a series that satisfies the hypotheses of (ii)$'$ but not (ii). (Thus, statement (ii)$'$ is stronger than statement (ii), since there are series that satisfy the hypotheses of (ii)$'$ but not (ii).)

(e) Show that the Ratio Test and "Enhanced Ratio Test" generalize to series $\sum_{n=1}^{\infty} a_n$ in which $a_n \neq 0$ for all $n$ sufficiently large.

B6. Find, with proof, all limit points of the subset of $A$ of $\mathbf{R}$ given by

$$A = \left\{ \frac{1}{n} + \frac{1}{m} : m, n \in \mathbf{N} \right\}.$$

(Your answer should be some subset $C \subseteq \mathbf{R}$ with the property that $x \in C$ if and only if $x$ is a limit point of $A$. Thus there will be two facts you'll need to prove about your set $C$: (a) if $x \in C$,then $x$ is a limit point of $A$, then $x \in C$, and (b) if $x$ is a limit point of $A$, then $x \in C$. You're going to find fact (b) much harder to prove than fact (a).)