MAA 4211, Fall 2021–Assignment 5's non-book problems (complete list)

Finish reading Section 4.4 before starting problems B4–B5.

B1. (a) Let $f : \mathbf{R} \to \mathbf{R}$ be a function. Prove that f is continuous if and only if for every open set $U \subseteq \mathbf{R}$, the set $f^{-1}(U)$ is open. (*Make sure you understand that* **nothing in the hypotheses or notation refers to "inverse function",** *so you cannot assume that the function f has an inverse.* For this problem, any argument that relies in any way on inverse functions is wrong, and completely misses the point.)

(b) Let $S \subseteq \mathbf{R}$. A subset $A \subseteq S$ is called *relatively open* (in S), or *open as a subset of* S, if for all $x \in A$, there exists $\epsilon > 0$ such that $V_\epsilon(x) \cap S$ is contained in A. For example, if $S = [0, 2]$, the interval $[0, 1)$ is relatively open in S (why?), even though it is not open in $\mathbf{R}$.

Let $f : S \to \mathbf{R}$ be a function. Prove that f is continuous if and only if for every open set $U \subseteq \mathbf{R}$, the set $f^{-1}(U)$ is relatively open in S. (The same comment about "inverse function" as in part (a) applies here as well.)

B2. Let $I \subseteq \mathbf{R}$ be a positive-length (i.e. non-singleton) interval and let $f : I \to \mathbf{R}$ be a continuous function. Show that $f(I)$ (the range of f) has the "between-ness property" used in the definition of *interval* given in class earlier this semester, and hence that $f(I)$ is itself an interval.

Note: The positive-length hypothesis is not necessary; it's just to spare you from having to deal with a trivial special case. If I is a singleton-set, then $f(I)$ is a singleton, hence trivially an interval.

B3. Let I and J be intervals, and $f : I \to J$ be a monotone function (see Abbott for definition).

(a) Show that if a is an endpoint of I that is contained in I, then $f(a)$ is an endpoint of J. Deduce from this that J contains exactly as many endpoints as I does (0, 1, or 2).

(b) Show that if f is surjective, then f is continuous.

> **Remark.** Given an interval I, a subset $A \subseteq \mathbf{R}$, and a function $f : I \to A$, consider the function $\hat{f} : I \to f(I)$ obtained from f by replacing the codomain A with $f(I)$ (i.e. define $\hat{f} : I \to f(I)$ by setting $\hat{f}(x) = f(x)$ for each $x \in I$). Problem B2 shows that if f is continuous, then its range $f(I)$ is an interval. Problem B3(b) says that for *monotone* functions f defined on an interval, the converse is true as well: if the range of f is an interval, then f is continuous. Thus, for a monotone real-valued function f whose *domain* is an interval, f is continuous if and only if its *range* is an interval.

(c) Assume now that $J = f(I)$ and that f is *strictly* monotone (strictly increasing or strictly decreasing[1]). Strict monotonicity implies that f is injective, and the assumption that $J = f(I)$ implies that f is surjective. Thus f is bijective, so an inverse function $f^{-1} : J \to I$ exists. Show that f^{-1} is strictly monotone (with the same "direction of monotonicity"—increasing or decreasing—as f) and continuous.

(d) For any interval I that contains a sub-interval of the form $[a, \infty)$ (with $a \in \mathbf{R}$), and any function $f : I \to \mathbf{R}$, we define "$\lim_{x\to\infty} f(x) = \infty$" to mean that for all $M \in \mathbf{R}$, there exists $K \in I$ such that all $x > K$ satisfy $f(x) > M$. The statement "$\lim_{x\to\infty} f(x) = -\infty$" is defined by replacing "$f(x) > M$" with "$f(x) < M$" above. For any interval I that contains a sub-interval of the form $(-\infty, b]$ (with $b \in \mathbf{R}$), and any function $f : I \to \mathbf{R}$, we define "$\lim_{x\to-\infty} f(x) = \infty$" and "$\lim_{x\to-\infty} f(x) = -\infty$" analogously, just replacing "$x > K$" with "$x < K$."

(i) Assume that $f : [0, \infty) \to \mathbf{R}$ is continuous, strictly increasing, and satisfies $\lim_{x\to\infty} f(x) = \infty$. Prove that f maps $[0, \infty)$ bijectively to $[f(0), \infty)$; i.e. that the map $\hat{f} : [0, \infty) \to [f(0), \infty)$ is a bijection. Then, using an earlier problem-part, then deduce the existence and continuity of an inverse function $\hat{f}^{-1}$.

(ii) Assume that $f : \mathbf{R} \to \mathbf{R}$ is continuous, strictly increasing, and satisfies both $\lim_{x\to\infty} f(x) = \infty$ and $\lim_{x\to-\infty} f(x) = -\infty$. Prove that f is a bijection. Then, using an earlier problem-part, deduce the existence and continuity of an inverse function f^{-1}.

(e) Define a collection of functions $\{f_n : n \in \mathbf{N}\}$ as follows: if $n \in \mathbf{N}$ is odd, define $f_n : (-\infty, \infty) \to (-\infty, \infty)$ by $f_n(x) = x^n$; if $n \in \mathbf{N}$ is even, define $f_n : [0, \infty) \to [0, \infty)$ by $f_n(x) = x^n$. Show that $\lim_{x\to\infty} f_n(x) = \infty$ and that, when n is odd, $\lim_{x\to-\infty} f_n(x) = -\infty$.

(f) For each $n \in \mathbf{N}$, let f_n be as in part (e).

(i) Show that for n odd, $f_n : \mathbf{R} \to \mathbf{R}$ is a bijection, and has a continuous, strictly increasing inverse function.

(ii) Show that for n even, $f_n : [0, \infty) \to [0, \infty)$ is a bijection, and has a continuous, strictly increasing inverse function.

Remark. For each $n \in \mathbf{N}$, the function $(f_n)^{-1}$ is , by definition, the n^{th}*-root* function, $x \mapsto x^{1/n} := (f_n)^{-1}(x)$. Thus, the n^{th}-root function is continuous and strictly increasing. The facts established above show that for odd n, every $x \in \mathbf{R}$ has a unique n^{th} root, while for even n, every $x \in [0, \infty)$ has a unique *non-negative* n^{th} root.

[1] "Strictly increasing" means that for all $x, y \in I$, if $x < y$ then $f(x) < f(y)$; "strictly decreasing" means that for all $x, y \in I$, if $x < y$ then $f(x) > f(y)$.

Furthermore, since the function $h : (-\infty, 0] \to [0, \infty)$ defined by $h(x) = -x$ is a bijection, and the composition of bijections is a bijection, when n is even the function $\tilde{f}_n = f_n \circ h : (-\infty, 0] \to [0, \infty)$ is a bijection, and $\tilde{f}_n(x) = f_n(-x) = (-x)^n = x^n$. Hence, for even n, every $x \in [0, \infty)$ also has a unique *non-positive* n^{th} root. It follows that, when n is even, every $x \in (0, \infty)$ has exactly two n^{th} roots, namely $x^{1/n}$ and $-x^{1/n}$.

B4. We say that a real-valued function is *bounded* if its range is a bounded subset of $\mathbf{R}$. (Note that our definition of "bounded sequence in $\mathbf{R}$" is a special case of this more-general definition: a bounded sequence in $\mathbf{R}$ is a bounded function from $\mathbf{N}$ to $\mathbf{R}$.)

(a) Let $A \subseteq \mathbf{R}$ be a nonempty, bounded interval, and let $f : A \to \mathbf{R}$ be a uniformly continuous function. Prove that f is bounded.

Note: This result is a special case of part (b), so if you're able to do part (b), you don't need to give a separate argument for part (a). Part (a) can be done in at least three ways, two of which are of no use in part (b), so the former is not really a warm-up for the latter. I've kept both parts of the problem since the "no use in part (b)" approaches to (a), if you happen to think of them, are still worthwhile for developing an understanding of uniform continuity.

(b) Same as part (a), but with "interval" replaced by "set". *Suggestion*: proof by contradiction.

B5. Prove that the composition of uniformly continuous is uniformly continuous. More precisely: let $A, B \subseteq \mathbf{R}$ and let $f : A \to \mathbf{R}$ and $g : B \to \mathbf{R}$ be uniformly continuous functions for which $f(A) \subseteq B$. Prove that that $g \circ f : A \to \mathbf{R}$ is uniformly continuous.

B6 (**"Bounded monotone functions have limits"**). (a) Let $a, b \in \mathbf{R}$, with $a < b$, let I be the open interval (a, b), and let $f : I \to \mathbf{R}$ be a bounded monotone function. Prove that $\lim_{x\to a+} f(x)$ and $\lim_{x\to b-}$ exist. *Hint*: Use the sequential characterization of limits of functions.

(b) Prove that the result of part (a) remains true if we replace a by the symbol $-\infty$ (in which case we replace "$\lim_{x\to a+} f(x)$" by "$\lim_{x\to -\infty} f(x)$") and/or replace b by the symbol ∞ (in which case we replace "$\lim_{x\to b-} f(x)$" by "$\lim_{x\to\infty} f(x)$").

B7. Prove the following lemma:

Lemma ("Substitution lemma for limits of functions"). Let $B, C \subseteq \mathbf{R}$, and let b and c be limit points of B and C respectively. Let f be a real-valued function defined on $B\backslash\{b\}$, and let g be a real-valued function defined on $C\backslash\{c\}$ to $\mathbf{R}$. (For what you will be proving, it doesn't matter whether f is defined at b, or whether g is defined at c, so we are *permitting* f and g to be defined at b and c, respectively, but not *requiring* them to be defined there.) Assume that

$f(B\backslash\{b\}) \subseteq C\backslash\{c\}$, that $\lim_{x\to b} f(x) = c$, and that $\lim_{y\to c} g(y)$ exists. Prove that

$$\lim_{x\to b} g(f(x)) = \lim_{y\to c} g(y).$$

Loosely speaking, the lemma says that, under the given hypotheses, we can evaluate $\lim_{x\to b} g(f(x))$ by making the substitutions "$f(x) = y$" and substituting "$y \to c$" for "$x \to b$" (the latter substitution being motivated by "$f(x) \to c$ as $x \to b$.") As a sample application of this lemma, using facts about the sine function that we have not discussed this semester: since $\lim_{y\to 0} \frac{\sin y}{y}$ exists (and equals 1), and $\lim_{x\to 3}(x-3)^2 = 0$, and $(x-3)^2 \in \mathbf{R} \setminus \{0\}$ when $x \in \mathbf{R} \setminus \{3\}$,

$$\lim_{x\to 3} \frac{\sin\left((x-3)^2 \right)}{(x-3)^2} = \lim_{y\to 0} \frac{\sin y}{y} = 1.$$

B8. Given a subset $A \subseteq \mathbf{R}$ and a point $x_0 \in A$, recall that we call x_0 an *interior point* of A if for some $\epsilon > 0$, the ϵ-neighborhood $V_\epsilon(x_0)$ is contained in A. The *interior of* A is defined to be the set of all interior points of A. We will use the notation A° for the interior of A. (You may also put the circle directly on top of the A. This is a bit tricky to do in LaTeX.) Note that $A^\circ \subseteq A$, by the definition above.

Let $A \subseteq \mathbf{R}$.

(a) Show that A° is an open set.

(b) Show that A° is the largest open set contained A, in the sense that if $B \subseteq A$ and B is an open set, then $A^\circ \supseteq B$.

(c) Show that $A^\circ = \bigcup\{B \subseteq \mathbf{R} : \ B \subseteq A \text{ and } B \text{ is an open set}\}$.

B9. Let $U \subseteq \mathbf{R}$, let $f : U \to \mathbf{R}$ be a function, and let $V \subseteq U$ be a nonempty subset.[2] As usual, we write $f|_V$ for the restriction of f to V. In parts (c) and (e) below, assume additionally that U is a union of positive-length intervals (note that "U is a positive-length interval" is a special case of this).

(a) Show that if f is continuous (respectively, uniformly continuous) then $f|_V$ is continuous (respectively, uniformly continuous).

(b) Show that if f is Lipschitz (see Abbott Exercise 4.4.9), then so is $f|_V$.

(c) Show that if f is differentiable, then so is $f|_V$.

[2]For what you're asked to do in this problem, nonemptiness of V is not essential; we can phrase definitions of continuity and differentiability in such a way that empty functions are automatically continuous and differentiable. In this problem, I just didn't want you spending time on the trivial empty-function cases.

(d) Assume now that V is *open*. Show that f is continuous at every point of V *if and only if* $f|_V$ is continuous. (One direction of this implication is immediate from part (a), without even using openness of V. All that's left to do is to prove is that the converse implication holds when V is open. To help understand what the issue is, consider the function $f : \mathbf{R} \to \mathbf{R}$ defined by $f(x) = \begin{cases} 5 & \text{if } x < 1, \\ 7 & \text{if } x \geq 1. \end{cases}$ Let $V_1 = (-\infty, 1)$ and $V_2 = [1, \infty)$. Then each of the functions $f|_{V_1} : V_1 \to \mathbf{R}$ and $f|_{V_2} : V_2 \to \mathbf{R}$ is *constant*, hence continuous, and $\mathbf{R} = \text{domain}(f) = V_1 \cup V_2$. But f is not continuous.)

(e) Again assume that V is *open*. Show that f is differentiable at every point of V *if and only if* $f|_V$ is differentiable. In the latter case, show that $(f')|_V = (f|_V)'$.

(To clarify the second item you're being asked to prove: there are two operations involved here, restriction-to-V and differentiation. You're being asked to show that if V is open, then these two operations commute with each other—i.e. that if you differentiate f on its domain U, and then restrict the derivative f' to V, you get the same function as if you had first restricted f to V, and then differentiated.)

10. As an application of some parts of B9, suppose you are given a function $f : \mathbf{R} \to \mathbf{R}$ defined by an equation of one of the following forms:

$$f(x) = \begin{cases} g_1(x) & \text{if } x \geq c, \\ g_2(x) & \text{if } x < c; \end{cases} \tag{1}$$

or

$$f(x) = \begin{cases} g_1(x) & \text{if } x > c, \\ g_2(x) & \text{if } x \leq c; \end{cases} \tag{2}$$

or

$$f(x) = \begin{cases} h(x) & \text{if } x \neq c, \\ k & \text{if } x = c; \end{cases} \tag{3}$$

or

$$f(x) = \begin{cases} h_1(x) & \text{if } x > c, \\ k & \text{if } x = c, \\ h_2(x) & \text{if } x < c; \end{cases} \tag{4}$$

Above, $c, k \in \mathbf{R}$ are some given real numbers, and $g_1, g_2 : \mathbf{R} \to \mathbf{R}$ and $h : \mathbf{R} \setminus \{c\} \to \mathbf{R}$, $h_1 : (c, \infty) \to \mathbf{R}$, and $h_2 : (-\infty, c) \to \mathbf{R}$ are some given functions. (I've written forms (3) and (4) separately since the different *appearance* of (3) and (4) may hinder some students from understanding that, modulo naming the functions, these two forms are completely equivalent to each other. For similar reasons, I've included forms (1) and (2) explicitly, even though they are special cases of form (4).)

For a given $x_0 \in \mathbf{R}$, consider the question of whether f is differentiable at x_0.

(a) Assume $x_0 > c$. If f is defined by equation (1) (respectively, equation (2), (3), (4)), show that f is differentiable at x_0 if and only if g_1 (respectively g_1, h, h_1) is differentiable at x_0, in which case $f'(x_0) = g_1'(x_0)$ (respectively $g_1'(x_0), h'(x_0), h_1'(x_0)$).

(b) Same as part (a), but for $x_0 < c$, and with various function-names changed accordingly.

(c) Assume that $x_0 = c$, and for further simplicity, assume that g_1, g_2, h, h_1, and h_2 are differentiable. (Remember that this means that each of these functions is differentiable at every point of its domain.)

(i) If f is defined by equation (3), write down the limit whose existence, *by definition*, determines whether f is differentiable at x_0.

(ii) If f is defined by (1), (2), or (4), then write down the pair of one-sided limits (in terms of the appropriate functions g_1, g_2, h_1, and/or h_2, and/or the number k) that, *by definition*, must exist for f to be differentiable at x_0. Under what additional (necessary and sufficient) condition on this pair of one-sided limits is f differentiable at x_0?

Note: If a derivative of any of the functions g_1, g_2, h_1, and/or h_2 appears explicitly in the limit(s) you write down in (i) or (ii) , your answer is wrong.

(d) Hypotheses as in (c), but assume additionally that $\lim_{x\to c} h'(x)$ exists, and that the one-sided limits $\lim_{x\to c+} g_1'(x), \lim_{x\to c-} g_2'(x), \lim_{x\to c+} h_1'(x), \lim_{x\to c-} h_2'(x)$ exist. How, if at all, do your answers to part (c) change?[3]

B11. Let $I \subseteq \mathbf{R}$ be an positive-length interval, let $x_0 \in I$, and let $f : I \to \mathbf{R}$ be a function that is continuous on I and differentiable on $I \setminus \{x_0\}$.

Below, **be careful not to assume that f has any additional properties**. For example, don't assume that f' is continuous on $I \setminus \{x_0\}$.

(a) Assume x_0 is an endpoint of I. If x_0 is a left endpoint of I, assume that $\lim_{x\to x_0^+} f'(x)$ exists; if x_0 is a right endpoint of I, assume that $\lim_{x\to x_0^-} f'(x)$ exists. Prove that f is differentiable at x_0 and that $f'(x_0)$ has the same value as the corresponding limit above (and hence f' not only exists at x_0 but is continuous there).

Hints: (1) The Mean Value Theorem is a great theorem! (2) The sequential characterization of limits of functions is also a good tool to have in your kit.

(b) Assume that x_0 is an interior point of I, and that $\lim_{x\to x_0} f'(x)$ exists. Prove that f is differentiable at x_0 and that $f'(x_0)$ has the same value as this limits (and hence, again that f' not only exists at x_0 but is continuous there).

Hint. Use part (a) on two appropriate subintervals of I, and apply the result of Abbott Exercise 4.2.10(b).

(c) Exhibit a function $f : \mathbf{R} \to \mathbf{R}$ having all of the following properties:

(i) f is differentiable at every point *other than* 0.

[3] *Answer*: They don't.

(ii) $\lim_{x\to 0} f'(x)$ exists; and

(iii) f is NOT differentiable at 0.

(*Hint*: If you've done—or even just read through—all the earlier non-book problems on this assignment, you've seen such a function, except for one insignificant change.)

Explain why this example does not contradict part (a).

B12. Let $I \subseteq \mathbf{R}$ be a positive-length interval, and let $f : I \to \mathbf{R}$ be a function.

(a) Prove that if f is differentiable, and the function $f' : I \to \mathbf{R}$ is bounded, then f is Lipschitz. (See Abbott Exercise 4.4.9 for the definition of "Lipschitz".)

(b) Prove that if I is closed and bounded, and f is continuously differentiable, then f is Lipschitz.

(c) We call f *locally Lipschitz* if for all $x \in I$ there exists $\delta > 0$ such that the restriction of f to the neighborhood $V_\delta(x) \cap I$ is Lipschitz.

Prove that if f is *continuously differentiable* (definition: f differentiable, and its derivative $f' : I \to \mathbf{R}$ is continuous), then f is locally Lipschitz.

B13. In this problem, you may assume that the sine and cosine functions have their usual trigonometric properties, as well as that the derivative of sine is cosine. You may also assume that $\pi > 3$.

Prove that $\sin x < x$ for all $x > 0$. (*Hint*: MVT.)