MAA 4212—Exercises on the Mean Value Theorem and related theorems—Spring 2002

All of the exercises below make use of the Mean Value Theorem or its corollaries, in one form or another, but some require you to use other theorems in addition.

1. Prove that

$$\frac{x}{1+x^2} < \tan^{-1} x < x, \ \forall x > 0.$$

(Here "$\tan^{-1}$" is the inverse tangent function, also called "arctan".)

2. Using induction, prove that a (real) polynomial of degree n can have at most n distinct real roots. (Don't use the Fundamental Theorem of Algebra to do this problem. The Fundamental Theorem of Algebra says that over the *complex* numbers, any polynomial can be factored essentially uniquely into linear terms. It's much deeper and much harder to prove than the problem I'm asking you to do.)

3. Let $a, b \in \mathbf{R}$. Prove that the equation $x^3 - ax + b = 0$ has exactly one real root if $a < 0$, and exactly three real roots if $4a^3 - 27b^2 > 0$.

4. Prove that, for all $x > 0$,

$$\text{(a)} \quad \sin x < x,$$

$$(a) \quad \cos x > 1 - \frac{x^2}{2},$$

$$(b) \quad x - \frac{x^3}{3!} < \sin x < x - \frac{x^3}{3!} + \frac{x^5}{5!}.$$

(Warning: if you try to use Taylor's Theorem, don't forget that numbers of the form "$\sin c$" or "$\cos c$" can be negative as well as positive!)

5. Your calculator can probably handle numbers with at least 10 significant digits, ranging in absolute value from from 10^{-99} to 10^{99}; thus there are roughly $10^{10} \cdot 199 \cdot 2 \approx 4 \times 10^{12}$ numbers you could potentially punch in and ask it to take sin or cos of. When you do ask it for sin or cos of something, it thinks for a bit, and then gives you the answer to at least 10 significant digits. It's not feasible for your calculator to have a table where it's stored the answers to every one of these 4×10^{12} possible questions. Figure out a plausible explanation of how the designer was able to get the sin and cos buttons to work so well.