MAA 4212, Spring 2002—Homework # 3 non-book problems

Hand both of these in.

B1. Let $I \subset \mathbf{R}$ be an open interval, and let $f : I \to \mathbf{R}$ be continuous. Suppose that f is continuously differentiable everywhere except perhaps at x_0. Prove that if $\lim_{x\to x_0} f'(x)$ exists, and has the value A, then $f'(x_0)$ exists and equals A. (Hence f is continuously differentiable at x_0.)

Remarks. (i) Note that this problem does not contradict the fact that, in general, existence of the derivative does not imply continuity of the derivative. (ii) An example in which this result is useful is the function $f : \mathbf{R} \to \mathbf{R}$ given by $f(x) = \begin{cases} 1 - \cos x, & x \leq 0 \\ x^{3/2}, & x > 0 \end{cases}$ Clearly this function is continuous at 0 and everywhere else, and clearly its derivative exists for $x \neq 0$ and is continuous for $x \neq 0$. Furthermore, the limit of $f'(x)$ as x approaches zero from either side exists, and both one-sided limits are equal (to 0). Using theorems proven to date in class, you still cannot immediately conclude that $f'(0)$ exists without going back to the definition of derivative, which is a pain. The result in problem 1 above tells you that $f'(0) = 0$ without any further work.

B2. Reconcile the previous problem with the phenomena illustrated by (a) the function $x \mapsto |x|$, from $\mathbf{R}$ to $\mathbf{R}$, and (b) the function in problem 1b on p. 108. In each case state whether the derivative exists at $x = 0$, whether the limit of the derivative exists at $x = 0$, whether the derivative is continuous at $x = 0$, and why problem B1 does not apply. (Regarding 108/1b, which you handed in previously, you do not need to rewrite all the details; just restate the relevant conclusions.)