MAA 4212, Spring 2002—Homework # 4

1. In class we proved the "alternating series test" theorem: if the real-valued sequence $\{a_n\}$ strictly alternates in sign, and $|a_n|$ decreases *monotonically* to zero, then $\sum a_n$ converges. Give a counterexample showing that the monotonicity assumption in this theorem is crucial. (I.e. find a counterexample to the following statement: if the sequence $\{a_n\}$ strictly alternates in sign, and $\lim_{n\to\infty} a_n = 0$, then $\sum a_n$ converges.)

2. Here is a True/False test. Note that statement (a) has a hypothesis that is missing in statements (b) and (c).

(a) If $\{a_n\}$ is a sequence of *non-negative* real numbers, and $\sum_n a_n$ converges, then $\sum_n a_n^2$ converges.

(b) If $\{a_n\}$ is a sequence of real numbers and $\sum_n a_n$ converges, then $\sum_n a_n^2$ converges.

(c) If $\{a_n\}$ is a sequence of real numbers and $\sum_n a_n$ converges, then $\sum_n a_n^3$ converges.

Take this True/False test and prove your answers. You will probably find (b) a little more difficult than (a). You will probably find (c) several orders of magnitude more difficult than (a) or (b). Think of (c) as extra credit rather than as a problem you are expected to be able to solve.

3. Let $\{a_{(m,n)} \mid (m,n) \in \mathbf{N}\times\mathbf{N}\}$ be a "doubly indexed sequence"—a map $A : \mathbf{N}\times\mathbf{N} \to \mathbf{R}$, where $a_{(m,n)} = A(m,n)$. It is sometimes useful to picture $\{a_{(m,n)}\}$ as an "infinity-by-infinity matrix". In this problem we are interested in attaching meaning to the notation "$\sum_{m,n} a_{(m,n)}$," also written "$\sum_{m,n=1}^{\infty} a_{(m,n)}$".

Definition. The doubly-indexed series $\sum_{m,n} a_{(m,n)}$ is *absolutely convergent* (or *converges absolutely*) if there exists a bijection $f : \mathbf{N} \to \mathbf{N}\times\mathbf{N}$ such that $\sum_{j=1}^{\infty} a_{f(j)}$ is absolutely convergent. (Said more loosely, we are calling the doubly-indexed series is absolutely convergent if there is some order in which we can add up the entries of the "infinite matrix" $\{a_{(m,n)}\}$ as the terms of an absolutely convergent singly-indexed series.)

(a) Prove that if $\sum_{m,n} a_{(m,n)}$ converges absolutely and $f, g : N \to \mathbf{N}\times\mathbf{N}$ are bijections, then $\sum_{j=1}^{\infty} a_{f(j)} = \sum_{j=1}^{\infty} a_{g(j)}$. Hence if $\sum_{m,n} a_{(m,n)}$ converges absolutely, we can unambiguously define

$$\sum_{m,n} a_{(m,n)} = \sum_{j=1}^{\infty} a_{f(j)}$$

where f is *any* bijection $\mathbf{N} \to \mathbf{N}\times\mathbf{N}$.

(b) Explain why we should not attach any numerical value (in $\mathbf{R}$) to the notation "$\sum_{m,n} a_{(m,n)}$" if this doubly-indexed series is *not* absolutely convergent.

(c) What is the most general condition on $\{a_{(m,n)}\}$ you can think of for which it would make sense to make the definition "$\sum_{m,n} a_{(m,n)} = \infty$"? Try to express your condition as a potentially testable criterion—think of an example in which you would want to say "$\sum_{m,n} a_{(m,n)} = \infty$" and see whether you can tell, from your criterion, whether that statement is true.

(d) Prove that if $\sum_{m,n} a_{(m,n)}$ is absolutely convergent then $\sum_{m=1}^{\infty} a_{(m,n)}$ converges for all

$n \in \mathbf{N}$, $\sum_{n=1}^{\infty} a_{(m,n)}$ converges for all $m \in \mathbf{N}$, and

$$\sum_{m,n} a_{(m,n)} = \sum_{m=1}^{\infty}(\sum_{n=1}^{\infty} a_{(m,n)}) = \sum_{n=1}^{\infty}(\sum_{m=1}^{\infty} a_{(m,n)}).$$

(e) Let $\sum_{n=1}^{\infty} b_n, \sum_{n=1}^{\infty} c_n$ be absolutely convergent. Prove that $\sum_{m,n} b_m c_n$ is absolutely convergent, and that

$$\sum_{m,n} b_m c_n = (\sum_{n=1}^{\infty} b_n)(\sum_{n=1}^{\infty} c_n).$$

Remark. In the absolutely convergent case, enumerating $\mathbf{N} \times \mathbf{N}$ in the order

$$\begin{array}{lll} (1,1) & & \\ (1,2) & (2,1) & \\ (1,3) & (2,2) & (3,1) \\ & & \cdots \end{array}$$

leads us to

$$\sum_{m,n} a_{(m,n)} = \sum_{k=1}^{\infty} \left(\sum_{n+m=k} a_{(m,n)} \right). \tag{1}$$

One of the main reasons that the conclusions above are important are in their application to power series (in which case we index ther terms using $\mathbf{N} \bigcup \{0\}$ rather than $\mathbf{N}$, but clearly this makes no difference in the conclusions above). Suppose you are multiplying two polynomials together, say $a_0 + a_1 x + \ldots + a_N x^N$ (i.e. $\sum_{n=0}^{N} a_n x^n$) and $b_0 + b_1 x + \ldots + b_M x^M$ (i.e. $\sum_{m=0}^{M} b_m x^m$). After multiplying out, you generally rewrite the result by grouping together all the terms with a given power of x, which is the finite-series statement

$$\left(\sum_{n=0}^{N} a_n x^n \right) \left(\sum_{m=0}^{M} b_m x^m \right) = \sum_{k=0}^{N+M} \left(\sum_{n+m=k} a_n b_m \right) x^k.$$

Since power series are absolutely convergent on their open intervals of convergence, parts (a) and (e) imply that on the smaller of the two open intervals of convergence of two power series, you can multiply power series together just as if they were polynomials (with infinitely many terms). For fun, you might try to show the identity $\sin^2 x + \cos^2 x = 1$ or $\sin x \cos x = \frac{1}{2}\sin(2x)$ or $(e^x)^2 = e^{2x}$ this way.

Notation convention for the remaining problems. If a function $g : (a,b) \to \mathbf{R}$ extends continuously to $[a,b]$, then clearly the continuous extension is unique. Therefore for any function g on (a,b) with a continuous extension to $[a,b]$, we can unambiguously define $\int_a^b g(x)dx = \int_a^b \tilde{g}(x)dx$, where $\tilde{g}$ is the continuous extension of g. From now on we adopt this convention. (Thus, for example, $\int_0^1 \frac{\sin x}{x}\, dx$ is defined even if we don't trouble ourselves to define the integrand at $x = 0$.)

In the problems below, You are allowed to use your knowledge of trigonometric functions and their derivatives, and to use the integration-by-parts formula you derived in HW problem p. 133/#17.

4. Suppose $g : (0, \pi) \to \mathbf{R}$ is differentiable and has bounded first derivative. Prove that

$$\lim_{n\to\infty} \int_0^\pi g(x) \sin(nx)\ dx = 0.$$

(By Problem 7 on the your recent exam, g extends to a continuous function on $[0, \pi]$, and hence so does $g(x) \sin nx$; thus the notation-convention above applies.)

5. In class we saw that $\sum_{n=1}^\infty 1/n^p$ converges if $p > 1$ but didn't try to evaluate the sum. In this problem you will end up computing the actual value of $\sum 1/n^2$ (by roundabout means).

In this problem, you are free to use the conclusion of the previous problem.

(a) Let $f : [0, \pi] \to \mathbf{R}$ be a function. Suppose f'' exists and is continuous on $[0, \pi]$, and that $f(0) = f(\pi) = 0$. For $0 < x < \pi$, define $g(x) = f(x)/\sin(x)$. Prove that limit of g' exists at both endpoints of $[0, \pi]$, and hence that g' extends to a continuous and (and therefore bounded) function on $[0, \pi]$. (Note: this problem is one reason I changed "Real mathematicians *don't* use L'Hôpital's Rule" into "Real mathematicians *rarely* use L'Hôpital's Rule".)

(b) Let f be as in part (a). Prove that

$$\lim_{n\to\infty} \int_0^\pi f(x) \frac{\sin(nx)}{\sin(x)}\ dx = 0.$$

(c) Verify that if n is any integer, then

$$\int_0^\pi x(\pi - x) \cos(2nx)\ dx = \begin{cases} -\pi/(2n^2), & n \neq 0 \\ \pi^3/6, & n = 0 \end{cases}.$$

(Note: for $n \neq 0$ the computation is simpler if you do not break the integral up into two pieces, one for $x^2 \cos 2nx$ and $x \cos 2nx$.) Use this to prove that

$$\sum_{n=1}^\infty \left(\int_0^\pi x(\pi - x) \cos(2nx)\ dx \right) = -\frac{\pi}{2} \sum_{n=1}^\infty \frac{1}{n^2}.$$

(d) Show that for all integers $n \geq 1$,

$$\cos(2x) + \cos(4x) + \cos(6x) + \ldots + \cos(2nx) = \frac{1}{2} \left(\frac{\sin((2n+1)x)}{\sin(x)} - 1 \right).$$

Use this to prove that

$$\sum_{n=1}^\infty \left(\int_0^\pi x(\pi - x) \cos(2nx)\ dx \right) = -\frac{1}{2} \int_0^\pi x(\pi - x) dx.$$

(e) Using the work above, determine the exact value of $\sum_{n=1}^\infty \frac{1}{n^2}$.