MAA 4212, Spring 2002—Homework #6 non-book problems

Hand in only the second exercise below.

1. Prove the following version of Taylor's Theorem. The difference between this version and the ones proven in class is that this one uses the Taylor polynomial of the same degree of differentiability as the function, rather than one less. The cost of this apparent improvement is that one does not get any sort of formula for the remainder, but just knowledge of the size of the remainder. For simplicity, I've based everything at the origin, but you should be able to translate this to the more general statement if you ever need to.

Theorem. Let U be a ball in $\mathbf{R}^n$ centered at $\mathbf{0}$. Let f be an m times continuously differentiable function. For $\mathbf{x} \in B$, define

$$P_m(\mathbf{x}) = f(\mathbf{0}) + \sum_i x_i f_i(\mathbf{0}) + \frac{1}{2}\sum_{i,j} x_i x_j f_{ij}(\mathbf{0}) + \cdots + \frac{1}{m!}\sum_{i_1,i_2,\ldots i_m} x_{i_1}x_{i_2}\cdots x_{i_m} f_{i_1 i_2 \cdots i_m}(\mathbf{0}),$$

and

$$R_m(\mathbf{x}) = f(\mathbf{x}) - P_m(\mathbf{x}).$$

(Above, f_{ij} means $\partial^2 f/\partial x_i \partial x_j$ etc, and a sum over k indices means a k-fold sum.) Prove that

$$\lim_{\mathbf{x}\to\mathbf{0}} \frac{R_m(\mathbf{x})}{\|\mathbf{x}\|^m} = 0.$$

Remark. Using the "big-oh, little-oh" terminology we introduced once in class, this says that $R_m(\mathbf{x})$ is $o(\|\mathbf{x}\|^m)$ ("little-oh of $\|\mathbf{x}\|^m$") as $\mathbf{x} \to 0$. Thus, if a function is m times continuously differentiable at a point, then its mth-order Taylor polynomial at that point is a good approximation to order m.

2. p. 214/ 16. This is a generalization of the "second partials test" you learned in Calculus III. To do the problem you will (probably) need problem 1 above, and the following definition.

Definition. Let A be an $n \times n$ symmetric matrix of real numbers, and for all $\mathbf{x} \in \mathbf{R}^n$, let $h_A(\mathbf{x}) = \sum_{i,j} A_{ij} x_i x_j$. (This is called a *quadratic form.*) We say that A is

positive-definite	if	$h_A(\mathbf{x}) > 0,\ \forall \mathbf{x} \neq \mathbf{0}$
positive-semidefinite	if	$h_A(\mathbf{x}) \geq 0,\ \forall \mathbf{x}$
negative-definite	if	$h_A(\mathbf{x}) < 0,\ \forall \mathbf{x} \neq \mathbf{0}$
negative-semidefinite	if	$h_A(\mathbf{x}) \leq 0,\ \forall \mathbf{x}.$

We also call A *definite* if A is either positive-definite or negative-definite, *semidefinite* if A is either positive semidefinite or negative semidefinite, and *indefinite* if A is neither positive-semidefinite nor negative-semidefinite. The identity matrix is an example of a matrix that is positive-definite; minus the identity is an example of a matrix that is negative-definite. The matrix $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ is positive-semidefinite but not positive-definite, and the matrices $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ are indefinite. *Warning:* One can show that a

2×2 symmetric matrix is definite iff its determinant is strictly positive (but the determinant alone does not tell you which of the two definite types the matrix is), and indefinite iff its determinant is strictly negative. However, for matrices of size 3×3 and up, *you cannot determine whether a matrix is definite from its determinant alone.*