MAA 4212—Improper Integrals

The Riemann integral, while perfectly well-defined, is too restrictive for many purposes; there are functions which we intuitively feel "ought" to be integrable, but which are not *Riemann* integrable according to the definition. For example, the expression

$$\int_0^1 \frac{1}{\sqrt{t}}dt$$

makes no sense as a Riemann integral, since the integrand is not defined at $t = 0$. Even if we fix that problem, by defining a function that's $t^{-1/2}$ for $t > 0$ and (say) 0 for $t = 0$, this new function is still not Riemann-integrable over $[0, 1]$ because it isn't bounded. However, if formally make the change of variables $t = u^2$ ("formally" means "shoot first, ask questions about validity later"), the integral above gets transformed into

$$\int_0^1 2\ du,$$

which is as nice an integral as they come. Furthermore, if we go back to our original integral and think of it as representing area under a curve, there is a useful sense in which this area is finite: take the area below the curve between $t = \epsilon$ and $t = 1$, and let $\epsilon \to 0$. Either way of looking at the original integral, the answer we *formally* calculate is 2. These considerations suggest that we ought to enlarge the class of functions we're willing to call "integrable", and modify our definition of "integral". The types of integrals we'll deal with in this handout are often called "improper integrals", but we'll simply call them "integrals" here.

Terminology. In this handout, the words "integral" and "integrable" will *not* be synonymous with "Riemann integral" and "Riemann-integrable". (In Rosenlicht, they are synonymous, but we will need to be clearer here on what notion of integration we're talking about.) We will use notation "Riemann $\int_a^b f(t)dt$" (with the word "Riemann" in front of the integral sign) to denote the Riemann integral. Whenever we write hypotheses such as "Let $f : [a, b] \to \mathbf{R}$", we understand this as short-hand for "Let $a, b \in \mathbf{R}$ with $a < b$ and let $f : [a, b] \to \mathbf{R}$;" analogous interpretations apply if $[a, b]$ is replaced by $(a, b]$, $[a, b)$, or (a, b). Also, we write "$\lim_{x\uparrow a}$" and "$\lim_{x\downarrow a}$" in place of $\lim_{x\to a-}, \lim_{x\to a+}$ respectively.

§1 Integrals over bounded intervals.

Definition 1. We will say that a real-valued function f is *GR-integrable* (for "generalized Riemann integrable") on the interval $[a, b]$ if either

(i) f is defined on $(a, b]$ and is Riemann-integrable over $[y, b]$ for all $y \in (a, b]$, and $\lim_{y\downarrow a}$ Riemann $\int_y^b f(t)dt$ exists; or

(ii) f is defined on $[a,b)$ and is Riemann-integrable over $[a,y]$ for all $y \in [a,b)$, and $\lim_{y \uparrow b}$ Riemann $\int_a^y f(t)dt$ exists.

(This is a temporary definition that will be generalized and finalized in Definition 3.)

Note that if f satisfies both conditions (i) and (ii), then it is Riemann-integrable over $[a,b]$. In particular, every Riemann-integrable function is GR-integrable.

Note. The terms "generalized Riemann integral" and "GR-integrable" are specific to these notes; they are not standard terminology.

Exercise.
1. Suppose $f : [a,b] \to \mathbf{R}$ is Riemann-integrable on $[a,b]$. Prove that for all $c \in [a,b]$, the functions g, h defined by $g(x) = \text{Riemann } \int_c^x f(t)dt, \quad h(x) = \text{Riemann } \int_x^c f(t)\, dt$ are continuous. (*Note for 2009 class: we essentially proved this in class; you do not need to do this exercise.*)

In particular, Exercise 1 implies that if f is Riemann-integrable on $[a,b]$, then

$$\lim_{y \downarrow a} [\text{Riemann } \int_y^b f(t)dt] = \text{Riemann } \int_a^b f(t)dt \tag{1}$$

and

$$\lim_{y \uparrow b} [\text{Riemann } \int_a^y f(t)dt] = \text{Riemann } \int_a^b f(t)dt. \tag{2}$$

This suggests using equations (1) and (2) to *define* the integral in certain non-Riemann-integrable cases.

Definition 2. Let $f : [a,b] \to \mathbf{R}$. If f satisfies condition (i) in Definition 1, we define the *generalized Riemann integral*

$$\int_a^b f(t)dt = \lim_{y \downarrow a} [\text{Riemann } \int_y^b f(t)dt]. \tag{3}$$

Similarly if f satisfies condition (ii) in Definition 1, we define

$$\int_a^b f(t)dt = \lim_{y \uparrow b} [\text{Riemann } \int_a^y f(t)dt]. \tag{4}$$

In both cases we will say that f is GR-integrable on $[a,b]$.

Note. In place of saying "f is GR-integrable", we often say "the integral exists" or "the integral converges".

Equations (1)–(2) show that there is no ambiguity in Definition 2; if f satisfies both (i) and (ii) in Definition 1, then the limits in equations (3) and (4) are equal. Moreover, if f is Riemann-integrable on $[c,d]$ then $\int_c^d f(t)dt = \text{Riemann } \int_c^d f(t)dt$. Hence if f is GR-integrable on $[a,b]$ we may write equations (3) and (4) more simply as

$$\int_a^b f(t)dt = \lim_{y \downarrow a} \int_y^b f(t)dt = \lim_{y \uparrow b} \int_a^y f(t)dt. \tag{5}$$

In the exercises below we will develop a useful comparison test for telling whether certain functions are GR-integrable. To prove the comparison test valid it is helpful to have the following simple lemma, which is an analog of (and is equivalent to) the Cauchy criterion for sequences. (This is essentially the Proposition on p. 74 of Rosenlicht, but it was phrased there in too narrow a way for our purposes.)

Lemma 1. Let $(E,d),(E',d')$ be metric spaces. Assume that p_0 is a cluster point of E, that E' is complete, and that f is a function from E to E'. Then $\lim_{p\to p_0} f(p)$ exists iff for all $\epsilon > 0$ there exists $\delta > 0$ such that $d'(f(p), f(q)) < \epsilon$ whenever $d(p,p_0)$ and $d(q,p_0)$ are both $< \delta$.

Proof. ($\Rightarrow$) Assume the limit exists and has value $L \in E'$. Let $\epsilon > 0$. Then there exists δ such that $d(p,p_0) < \delta$ implies $d'(f(p), L) < \epsilon/2$. Hence if $p, q \in B_\delta(p_0)$, the triangle inequality implies $d'(f(p), f(q)) < \epsilon$.

($\Leftarrow$) Let $\epsilon > 0$, and choose $\delta > 0$ such that $p, q \in B_\delta(p_0)$ implies $d'(f(p), f(q)) < \epsilon/2$. Since p_0 is a cluster point of E there exists a sequence $\{p_n\}$ converging to p_0. Let N be such that $n \geq N$ implies $p_n \in B_\delta(p_0)$. Then for $n, m \geq N$ we have $d'(f(p_n), f(p_m)) < \epsilon/2$, so the sequence $\{f(p_n)\}$ is Cauchy. Since E' is complete, this sequence converges, say to L. Let $n \geq N$ be such that $d'(f(p_n), L) < \epsilon/2$ and let $q \in B_\delta(p_0)$. Then $d'(f(q), L) \leq d'(f(q), f(p_n)) + d'(f(p_n), L) < \epsilon$, so $\lim_{q\to p_0} f(q)$ exists (and equals L). ∎

An equivalent form of this lemma is the first sentence of:

Lemma 1′. Let the hypotheses be as in Lemma 1. Then $\lim_{p\to p_0} f(p)$ exists iff for every sequence $\{p_n\}$ that converges to p_0, the sequence $f(p_n)$ converges. If $\lim_{p\to p_0} f(p)$ exists, then it equals $\lim_{n\to\infty} f(p_n)$ for every sequence $\{p_n\}$ that converges to p_0.

Remark. We have used a weaker version of this lemma before: $\lim_{p\to p_0} f(p)$ exists iff there exists L such that for every sequence $\{p_n\}$ that converges to p_0, the sequence $f(p_n)$ converges <u>to</u> $\underline{L}$. (*Note to 2009 class: "We" in previous sentence means "people who have taken MAA 4211 with me".*) Lemma 1′ strengthens the "$\Leftarrow$" implication by removing the assumption that the sequences $\{f(p_n)\}$ have the same limit.

Proof of Lemma 1′. ("$\Rightarrow$" direction of first sentence of conclusion): Turn the crank.

("$\Leftarrow$" direction of first sentence of conclusion, and second sentence of conclusion) Suppose that whenever $p_n \to p_0$, $\{f(p_n)\}$ converges. Choose two such sequences $\{p_n\}, \{q_n\}$ and suppose $f(p_n) \to L_1$ while $f(q_n) \to L_2$. The "spliced" sequence $p_1, q_1, p_2, q_2, \ldots$ also converges to p_0, but if $L_1 \neq L_2$, the sequence $f(p_1), f(q_1), f(p_2), f(q_2), \ldots$ cannot converge. Hence $L_1 = L_2$; i.e for all sequences $\{p_n\}$ converging to p_0, the limiting value of $f(p_n)$ is the same, say L. Now suppose that $\lim_{p\to p_0} f(p) \neq L$ or doesn't exist. Then there exists $\epsilon > 0$ such that for all n, there exists $p_n \in B_{1/n}(p_0)$ such that $d'(f(p_n), L) \geq \epsilon$. Since $p_n \to p_0$, this is a contradiction. ∎

Exercises.

2. Let $b > a$. Prove that $f : x \mapsto \frac{1}{(x-a)^p}$ is GR-integrable on $[a, b]$ iff $p < 1$. (Here p can be any real number. Thus for $0 < p < 1$, f is GR-integrable but not Riemann-integrable on $[a, b]$.)

3. Let $b > a$. Assume f is Riemann-integrable on $[y, b]$ whenever $a < y \leq b$. (Note that any function continuous on $(a, b]$ satisfies this criterion, even if not defined or continuous at a.) Assume there exists a function $g : (a, b] \to \mathbf{R}$, GR-integrable on $[a, b]$, such that $|f(x)| \leq g(x)$, $\forall x \in (a, b]$. Prove that f is GR-integrable on $[a, b]$ and that $|\int_a^b f(x)dx| \leq \int_a^b g(x)dx$.

4. Let $b > a$. Assume f is continuous on $(a, b]$ and that, for some $p < 1$, the function $x \mapsto (x - a)^p f(x)$ is bounded on $(a, b]$. Prove that f is GR-integrable on $[a, b]$.

5. State the analogs of exercises 1-3 with the roles of a and b reversed (i.e. with $(a, b]$ replaced by $[a, b)$). Essentially the same proofs work, of course.

6. Without using any reference to trigonometric functions or their inverses, prove that $\int_0^1 \frac{1}{\sqrt{1-t^2}}dt$ exists (i.e. that $(1 - t^2)^{-1/2}$ is GR-integrable on $[0, 1]$). Remark: The value of this integral can be taken as the *definition* of $\pi/2$.

In this handout, we will refer to a point $x_0 \in \mathbf{R}$ as a *singularity* of f if x_0 is in the closure of the domain of f, but there exists no closed interval containing x_0 over which f is Riemann-integrable. To make the concept more concrete, it's useful to picture a function which is defined and continuous *near* x_0 but not *at* x_0, and for which $\lim_{x\to x_0} f(x)$ does not exist (e.g. 0 is a singularity of $x \mapsto 1/x$ on $[0, \infty)$ and on $(-\infty, \infty)$). Garden-variety functions to which Definitions 1 and 2 apply are functions that are continuous on the interior of an interval but have a singularity at one endpoint or the other, but not both. We next want to extend our definition of "GR-integrable on $[a, b]$" to include functions that have singularities at more than one point (e.g. both endpoints) and/or at an interior point of the interval of integration. Our extension will be based on the next exercise, which you should think of as a generalization of the Proposition at the bottom of p. 123 in Rosenlicht.

Exercise.

7. Prove that f is GR-integrable over $[a, b]$, in the sense of Definition 2, if and only if at least one of the following two conditions holds: (i) for all $c \in (a, b)$, f is GR-integrable over $(a, c]$ and Riemann-integrable over $[c, b]$, or (ii) for all $c \in (a, b)$, f is Riemann-integrable over $[a, c]$ and GR-integrable over $[c, b)$. When the integrals exist, prove that

$$\int_a^b f(t)dt = \int_a^c f(t)dt + \int_c^b f(t)dt. \tag{6}$$

(End of exercise 7.)

Now suppose we have a function f defined on $[a, b]$, continuous except for an interior singularity at a single point $c \in (a, b)$. If the GR-integrals over $[a, c]$ and $[c, b]$ exist, we can *define* the GR integral of f over $[a, b]$ by taking equation (6) as definition. Similarly, if f is continuous on the interior but singular at both endpoints, and if for some interior point c the GR integrals over $[a, c]$ and $[c, b]$ exist, then we can again take (6) as a definition of the left-hand side. Finally, if we have a function which is continuous $[a, b]$ except for singular points $s_1, \ldots, s_n$, we can chop up $[a, b]$ into a finite number of sub-intervals on which f has only one singularity (intersperse non-singular points y_i with the s_i's), and use the analog of equation (6) with one term for each sub-interval to define the left-hand side. Looking over what we've just said, we see that we never really needed f to be continuous off the set of singular points (though that's most commonly what we see in practice). Our formal definition becomes:

Definition 3. Suppose the real-valued function f has the following property: there exist points $s_0, s_1, \ldots, s_{n+1}$, with $a = s_0$ and $b = s_{n+1}$, such that f is defined at every point of $[a, b]$ except possibly the s_i's, and such that for $0 \le i \le n$, f is GR-integrable over $[s_i, s_{i+1}]$ in the sense of Definition 1. Then we say that f is GR-integrable over $[a, b]$, and define

$$\int_a^b f(t)dt = \sum_{i=0}^{n} \int_{s_i}^{s_{i+1}} f(t)dt. \tag{7}$$

(Note that we don't *require* f to be singular at the s_i's; we simply *allow* it. In general, to apply Definition 1 we'll have to intersperse nonsingular points between the singular points.)

There is a potential problem with this definition. In general, if f is GR-integrable over $[a, b]$, there will be infinitely many choices for the s_i. For example, if $f(x) = [x(1-x)]^{-1/2}$ on [0,1], we could choose s_1 to be *any* number strictly between 0 and 1. For the integral over $[0, 1]$ to be well-defined, we need to know that the right-hand side of (6) does not depend on where we put the non-singular s_i's.

Exercise.

8. Prove that for functions GR-integrable according to Definition 3, $\int_a^b f(t)dt$ is well-defined (i.e. does not depend on the choice of the points s_i).

§2 **Integrals over unbounded intervals.**

Next we want to allow for the possibility of integrating functions over infinite intervals (e.g. $[0, \infty)$). The most intuitive way to do this is the following.

Definition 4. Let $a \in \mathbf{R}$. We say f is GR-integrable on $[a, \infty)$ (or "$\int_a^\infty f(t)dt$

exists", or "$\int_a^\infty f(t)dt$ converges") iff (i) for every $y > a$, f is GR-integrable over $[a, y]$ (in the sense of Definition 3), and (ii) $\lim_{y\to\infty} \int_a^y f(t)dt$ exists. In the GR-integrable case, we define $\int_a^\infty f(t)dt$ to be the value of this limit. We define "GR-integrable over $(-\infty, a]$" and "$\int_{-\infty}^a f(t)dt$" similarly. We say f is GR-integrable on $(-\infty, \infty)$ if it is GR-integrable on $(-\infty, 0]$ and $[0, \infty)$, in which case we define $\int_{-\infty}^\infty f(t)dt = \int_{-\infty}^0 f(t)dt + \int_0^\infty f(t)dt$.

Exercises.

9. Prove that, in the definition of integrability over $(-\infty, \infty)$, the number "0" could have been replaced by any real number without changing the set of functions being called GR-integrable or (in the GR-integrable case) the value of the integral.

10. Let $a > 0$. Prove that $\int_a^\infty \frac{1}{x^p} dx$ exists iff $p > 1$. (Here p can be any real number.)

11. Determine all values of p for which $\int_0^\infty \frac{1}{x^p} dx$ exists.

12. Let $a \in \mathbf{R}$. Assume f is GR-integrable on $[a, y]$ for all $y > a$. Assume there exists a function $g : (a, \infty) \to \mathbf{R}$, GR-integrable on $[a, \infty)$, such that $|f(x)| \le g(x)$, $\forall x > a$. Prove that f is GR-integrable on $[a, \infty)$ and that $|\int_a^\infty f(x)dx| \le \int_a^\infty g(x)dx$.

13. Let $a \in \mathbf{R}$. Assume f is continuous on (a, ∞) and that, for some $p > 1$ and some $q < 1$, the function $x \mapsto ((x-a)^p + (x-a)^q)\, f(x)$ is bounded on (a, ∞). Prove that $\int_a^\infty f(t)dt$ exists.

14. Let $\epsilon > 0$. Let P be a polynomial function. Prove that, no matter how small ϵ is or how large the degree of P is, $\int_0^\infty e^{-\epsilon x} P(x)dx$ converges.

15. *Note to 2009 class: if you did not cover "*$\limsup$*" and "*$\liminf$*" in MAA 4211, ignore this exercise, and the Remark below it, for now.* Suppose f is defined on $[a, \infty)$ and $f(x) \ge 0$ $\forall x$. Prove that if $\int_a^\infty f(x)dx$ exists, then $\liminf_{x\to\infty} f(x) = 0$. (Note: previously we defined "lim inf" only for sequences, but you should be able to figure out how to extend the definition to the current situation.)

Remark. The statement in exercise 15 would be false if "$\liminf$" were replaced by "lim". First, $\lim_{x\to\infty} f(x)$ doesn't have to exist for $\int_a^\infty f(x)dx$ to exist. Second, it is even possible for the integral to converge if there is sequence $x_n \to \infty$ for which $f(x_n) \to \infty$. As an example, consider a function f which is zero most places, except for triangular spikes centered at the positive integers. For the spike centered at n, let the base of the spike have width 2^{-2n} and height $2 \cdot 2^n$, so that the triangle has area 2^{-n}. Then it's not hard to show that f is GR-integrable on $[0, \infty)$ and that the integral equals the (convergent) geometric series $\sum_1^\infty 2^{-n}$, even though $f(n) \to \infty$. If we drop the restriction that f be nonnegative, it is easy to come up with other examples of counterintuitive phenomena; see exercises 17-19.

§3 Change-of-Variables Formula.

Often a formal change of variables made to simplify the computation of an integral turns a "proper" integral into an "improper" one, or vice-versa. Sometimes a change of variables turns one "improper" integral into another. One needs to know whether such changes of variable are valid. For simplicity, we will state the theorem only for functions defined on an interval of the form $(a, b]$ or $[b, \infty)$ and which satisfy the corresponding parts of Definitions 1 or 4. A more general statement isn't hard to prove, but is messier to state.

Change-of-Variables Theorem. Suppose f is continuous on an interval I, where either (i) $I = (a, b]$ or (ii) $I = [b, \infty)$. Let $\phi : I \to \mathbf{R}$ be continuous on I and continuously differentiable on the interior of I. In case (i), suppose that

$$\lim_{t \downarrow a} \phi(t) = c,$$

where we allow the symbol c to stand for a real number or for ∞. (Thus we assume that either the limit exists, or that $\lim_{t \downarrow a} \phi(t) = \infty$.) Similarly, in case (ii), suppose that

$$\lim_{t \to \infty} \phi(t) = c,$$

where again c may stand for ∞. Then in each case ((i) and (ii)) either both of the integrals

$$\int_a^b f(\phi(t))\phi'(t)dt, \quad \int_c^{\phi(b)} f(x)dx \tag{8}$$

exist, or neither does. (If $c = \infty$ or $c \geq \phi(b)$, see the last remark at the end of these notes.) When the integrals exist, they are equal.

Proof. We will write out the proof only for the case in which $I = (a, b]$ and $c < \phi(b)$ is a real number; the other cases are similar.

Assume the second integral in (8) exists. Note that for $u > a$, the hypotheses of Corollary 3 on p. 128 of Rosenlicht (the change-of-variables formula for "proper" integrals) are satisfied with $U = (u, b)$. By hypothesis $\lim_{y \downarrow c} \int_y^{\phi(b)} f(x)dx$ exists and $\lim_{u \downarrow a} \phi(u) = c$. Hence

$$\int_c^{\phi(b)} f(x)dx = \lim_{y \downarrow c} \int_y^{\phi(b)} f(x)dx = \lim_{u \downarrow a} \int_{\phi(u)}^{\phi(b)} f(x)dx = \lim_{u \downarrow a} \int_u^b f(\phi(t))\phi'(t)dt.$$

Thus the limit on the extreme right, which is the definition of $\int_a^b f(\phi(t))\phi'(t)dt$, exists and equals the integral on the extreme left.

Conversely, suppose that the first integral in (8) exists. Then, as above, we have

$$\int_a^b f(\phi(t))\phi'(t)dt = \lim_{u \downarrow a} \int_u^b f(\phi(t))\phi'(t)dt = \lim_{u \downarrow a} \int_{\phi(u)}^{\phi(b)} f(x)dx.$$

Let $y_n \downarrow c$. Since $\lim_{t \downarrow a} \phi(t) = c$, the Intermediate Value Theorem implies that ϕ maps the interval $(a, a + \delta]$ onto $(c, \phi(a + \delta)]$ for all $\delta < b - a$. Thus there exists a sequence $u_n \to a$ such that $y_n = \phi(u_n)$ for all n. Continuing the chain of equalities above, we then have

$$\lim_{u \downarrow a} \int_{\phi(u)}^{\phi(b)} f(x)dx = \lim_{n \to \infty} \int_{\phi(u_n)}^{\phi(b)} f(x)dx = \lim_{n \to \infty} \int_{y_n}^{\phi(b)} f(x)dx.$$

Applying Lemma $1'$, $\lim_{y\downarrow c}\int_y^{\phi(b)} f(x)dx$ exists and equals the first integral in (8). ∎

Exercises.

16. Re-do exercise 10 by using exercise 1 and a change of variables.

17. Prove that if $f : [0,\infty) \to \mathbf{R}$ is a nonnegative, decreasing function, with $\lim_{x\to\infty} f(x) = 0$, then $\int_0^\infty f(x)\sin(x)dx$ converges.

18. Prove that $\int_0^\infty \sin(x)dx$ does not converge, but that $\int_0^\infty \sin(x^2)\ dx$ and $\int_0^\infty \sqrt{x}\sin(x^2)dx$ do converge. (To understand what is "enabling" the last two integrals to converge, even though the integrand is not approaching 0—and, in the last integral, is even unbounded—it is instructive to graph the integrands.)

19. Prove that $\int_0^1 \sin(\frac{1}{x})dx$ exists.

Final remarks.

1. From Definition 3 and equation (6) it is clear that "f GR-integrable on $[a,b]$" does not require the function f even to be defined at a and b. Even if $f(a)$ and $f(b)$ are defined, the values $f(a), f(b)$ affect neither GR-integrability (or Riemann integrability for that matter) nor the value of the integral. Therefore we define "GR-integrable on $(a,b]$," "GR-integrable on $[a,b)$," and "GR-integrable on (a,b)" all to mean the same thing as "GR-integrable on $[a,b]$." The terminology "GR-integrable on (a,b)" is the most flexible, since it allows for the cases $a=-\infty$ and $b=\infty$.

2. So far we have defined integrals $\int_a^b f(x)\ dx$ only when $a<b$. We extend our definition to allow for $a\geq b$ in the same way as for Riemann integrals:

   - In the generalized case if $a>b$ we say that $\int_a^b f(x)\ dx$ exists iff f is GR-integrable on (b,a), in which case we define $\int_a^b f(x)\ dx = -\int_b^a f(x)\ dx$. (We have used open-interval notation here in order include the cases in which a or b is infinite.)
   - For $a\in\mathbf{R}$ we declare every function to be GR-integrable over the "interval" $[a,a]$, with $\int_a^a f(x)\ dx = 0$.