MAA 4212—Some lecture notes for 1/21/09

Proposition. *Let $a, b \in \mathbf{R}, a < b$. A function $f : [a,b] \to \mathbf{R}$ is integrable on $[a,b]$ if and only if for all $\epsilon > 0$ there exist step functions $f_1, f_2 : [a,b] \to \mathbf{R}$ such that (i) $f_1(x) \le f(x) \le f_2(x)$ for all $x \in [a,b]$, and (ii) $\int_a^b f_2 - \int_a^b f_1 < \epsilon$.*

Proof: (partial). The "if" direction was proven cleanly in class. We prove just the "only if" direction below.

Assume f is integrable on $[a,b]$ and let $\epsilon > 0$. Then there exists $\delta > 0$ such that

$$U_\delta(f) - L_\delta(f) < \epsilon. \tag{1}$$

Select such a δ, and let $\mathcal{P} = \{x_j\}_{j=0}^N$ be a partition of $[a,b]$ of width less than δ. Since f is integrable on $[a,b]$, it is bounded on $[a,b]$ (by a previously proven proposition), and hence is bounded on each interval $[x_{j-1}, x_j], 1 \le j \le N$. For each such j, let M_j and m_j be the supremum and infimum, respectively, of $\{f(x) \mid x \in [x_{j-1}, x_j]\}$. Define $f_1, f_2 : [a,b] \to \mathbf{R}$ by

$$\begin{aligned} f_2(x) &= \begin{cases} M_j & \text{if } x_{j-1} < x < x_j \text{ for some } j, \\ f(x_j) & \text{if } x = x_j \text{ for some } j, \end{cases} \\ f_1(x) &= \begin{cases} m_j & \text{if } x_{j-1} < x < x_j \text{ for some } j, \\ f(x_j) & \text{if } x = x_j \text{ for some } j. \end{cases} \end{aligned}$$

Then f_1, f_2 are step functions on $[a,b]$, and $f_1(x) \le f(x) \le f_2(x)$ for all $x \in [a,b]$. It remains to show only that $\int_a^b f_2 - \int_a^b f_1 < \epsilon$.

In previous work, we calculated the integral of a general step function. Applying this calculation to f_1 and f_2, we have

$$\int_a^b f_2 = A_2 := \sum_{j=1}^N M_j(x_j - x_{j-1}) \tag{2}$$

and

$$\int_a^b f_1 = A_1 := \sum_{j=1}^N m_j(x_j - x_{j-1}). \tag{3}$$

(In case you haven't seen it before, ":=" is used to indicate that we are defining the notation to the left of the ":=" to mean whatever's to the right of the ":=".)

Now let $\mathcal{S}$ be the set of Riemann sums of f associated to $\mathcal{P}$. On each interval $[x_{j-1}, x_j]$ we have $m_j \le f(x) \le M_j$. Hence, for any pointing T of $\mathcal{P}$, $A_1 \le S(f; \mathcal{P}, T) \le A_2$. Thus A_1 and A_2 are, respectively, lower and upper bounds for $\mathcal{S}$.

We claim that A_2 is the *least* upper bound of $\mathcal{S}$. For suppose not. Then for some $\epsilon' > 0$, $A_2 - \epsilon'$ is an upper bound for $\mathcal{S}$. For $1 \le j \le N$, since $M_j = \text{LUB}\{f(x) \mid x \in [x_{j-1}, x_j]\}$, there exists $t_j \in [x_{j-1}, x_j]$ such that $f(t_j) > M_j - \epsilon'/(b-a)$. The collection $T = \{t_j\}_{j=1}^N$ is a pointing of $\mathcal{P}$, and

$$\begin{aligned} S(f;\mathcal{P},T) &> \sum_{j=1}^{N}(M_j - \frac{\epsilon'}{b-a})(x_j - x_{j-1}) \\ &= \sum_{j=1}^{N} M_j(x_j - x_{j-1}) - \frac{\epsilon'}{b-a}\sum_{j=1}^{N}(x_j - x_{j-1}) \\ &= A_2 - \epsilon', \end{aligned}$$

a contradiction since, by our definition, $A_2 - \epsilon'$ is an upper bound for $\mathcal{S}$.

Therefore $A_2 = \text{LUB}(\mathcal{S})$, which, by definition, is the number $S_U(f;\mathcal{P})$. A similar argument shows that $A_1 = S_L(f;\mathcal{P})$. Then, since $L_\delta(f) \le S_L(f;\mathcal{P}) \le S_U(f;\mathcal{P}) \le U_\delta(f)$, inequality (1) implies that

$$A_2 - A_1 = S_U(f;\mathcal{P}) - S_L(f;\mathcal{P}) \le U_\delta(f) - L_\delta(f) < \epsilon.$$

But by equations (2)–(3), $A_i = \int_a^b f_i$ for $i = 1, 2$, so we are done. ∎