MAA 4212, Spring 2017—Assignment 1's non-book problems

For the problems below, recall that if X is a set and (Y, d_Y) (henceforth abbreviated "Y") is a metric space, a function $f : X \to Y$ is called *bounded* if the range of f is a bounded subset of Y.

B1. Let X be a nonempty set and let (Y, d_Y) be a metric space. Let $B(X, Y)$ denote the set of all bounded functions from X to Y. (Note that if Y is a bounded metric space, then all functions $X \to Y$ are bounded, so $B(X, Y)$ is the set of *all* functions $X \to Y$ in this case.)

(a) Let $f, g \in B(X, Y)$. Show that $\{d_Y(f(x), g(x)) \mid x \in X\}$ is a bounded, nonempty set in $\mathbf{R}$, hence has a supremum in $\mathbf{R}$.

(b) In view of part (a), we can define a function $\tilde{D} : B(X, Y) \times B(X, Y) \to \mathbf{R}$ by

$$\tilde{D}(f, g) = \sup\{d_Y(f(x), g(x)) \mid x \in X\}.$$

Show that $\tilde{D}$ is a metric on $B(X, Y)$.

Terminology: $\tilde{D}$ is called the *uniform metric* on $B(X, Y)$.

Henceforth "$B(X, Y)$" **is short-hand for the metric space** $(B(X, Y), \tilde{D})$.

(c) Let (f_n) be a sequence in $B(X, Y)$. Prove that (f_n) converges in $B(X, Y)$ if and only if (f_n) converges uniformly, and that in the convergent case, the pointwise-limit function is the same as the metric-space-limit function. (Note: one ingredient of the proof is the result of problem 38 on p. 94 in Rosenlicht, provided that you assume in #38 that X is only a set, not a metric space. Do #38 first, with this generalized assumption, so that you can use the result here.)

(d) Prove that if Y is complete, then $B(X, Y)$ is complete.

(e) Show that $B(\mathbf{N}, \mathbf{R}) = \ell^\infty(\mathbf{R})$ (i.e. the underlying sets are the same, and the metrics are the same). Thus, part (d) above shows that $\ell^\infty(\mathbf{R})$ is complete—a fact you already know if you were in my section last fall and succeeded in doing problem B7(c) on Homework Assignment 5, or were in one of the other sections and succeeded in doing the last part of Problem 6.6 in Dr. McCullough's notes. (My "$\ell^\infty(\mathbf{R})$" is the "ℓ^∞" in Dr. McCullough's Problem 6.6.)

(f) For $n \in \mathbf{N}$, let $J_n = \{1, 2, \ldots, n\}$. Note that, for any n, *every* function $J_n \to \mathbf{R}$ is bounded. What is the relation between the metric spaces $B(J_n, \mathbf{R})$ and $(\mathbf{R}^n, d_\infty)$?

B2. Now let both X and Y be metric spaces, and let $BC(X, Y) \subset B(X, Y)$ denote the set of bounded *continuous* functions $X \to Y$. Let $\tilde{D}$ now denote the restriction to $BC(X, Y) \times BC(X, Y)$ of the uniform metric defined in problem B1.

(a) If X is compact, what is the relation between the metric space $(BC(X, Y), \tilde{D})$ and the metric space $(C(X, Y), D)$, where D is the metric on $C(X, Y)$ defined in class?

(b) Prove that if Y is complete, then $(BC(X,Y), \tilde{D})$ is complete.

Remark. In many areas of mathematics, an important question is: given a sequence of functions that have some nice properties, if that sequence converges (in whatever sense is of interest in context; there are several different notions of convergence), does the limit function have those same nice properties? Part of what B1(d) and B2(b) are asserting is that for the "nice properties" of boundedness (in the setting of B1), or boundedness and continuity (in the setting of B2), the answer is "yes" if our notion of convergence is uniform convergence or, equivalently (by B1(c)), convergence in the metric space $B(X,Y)$.