MAA 4212, Spring 2017—Assignment 2's non-book problems

B1. Let X and Y be metric spaces, $(f_n : X \to Y)_{n=1}^{\infty}$ a sequence of functions, and $f : X \to Y$ a function. Assume that there is a real-valued sequence $(c(n))_{n=1}^{\infty}$ such that (i) for all $n \in \mathbf{N}$ and $x \in X$, $d_Y(f_n(x), f(x)) \leq c(n)$, and (ii) $\lim_{n\to\infty} c(n) = 0$. Prove that (f_n) converges uniformly to f.

Thus, to prove that a sequence (f_n) converges uniformly to a given function f, it suffices to find, for each n, a uniform upper bound $c(n)$ on the distances $d_Y(f_n(x), f(x))$ (where "uniform" means "independent of x"), with the property that $c(n) \to 0$ as $n \to \infty$. In practice, this is virtually always how uniform convergence is shown. (See, for example, the handed-out solutions to Rosenlicht problem IV.34(a).)

Exercises on the Mean Value Theorem

All of the exercises below make use of the Mean Value Theorem (MVT) or its corollaries, in one form or another, but some require you to use other theorems in addition. You may assume that the trigonometric and inverse trigonometric functions have the derivatives you learned in Calculus I-II-III.

B2. Let $a, b \in \mathbf{R}, a < b$, and assume that $f, g : [a, b) \to \mathbf{R}$ are continuous, and are differentiable on (a, b). Assume also that $f(a) = g(a)$ and that $f'(x) > g'(x)$ for all $x \in (a, b)$. Prove that $f(x) > g(x)$ for all $x \in (a, b)$.

B3. Prove that

$$\text{(a)} \qquad \frac{x}{1+x^2} < \tan^{-1} x < x \quad \text{for all } x > 0,$$

and

$$\text{(b)} \qquad x < \sin^{-1} x < \frac{x}{\sqrt{1-x^2}} \quad \text{for } \ 0 < x < 1.$$

(Here "$\tan^{-1}$" and "$\sin^{-1}$" are the inverse tangent and inverse sine functions, also known as "arctan" and "arcsin" respectively.)

B4. Prove that, for all $x > 0$,

$$\text{(a)} \quad \sin x < x,$$

$$\text{(b)} \quad \cos x > 1 - \frac{x^2}{2}, \qquad \text{and}$$

$$\text{(c)} \quad x - \frac{x^3}{3!} < \sin x < x - \frac{x^3}{3!} + \frac{x^5}{5!} \ .$$

(Warning: if you try to use Taylor's Theorem, don't forget that numbers of the form "$\sin c$" or "$\cos c$" can be negative as well as positive!)

Just FYI: Once upon a time, problems like proving the inequality in B4(a), or the right-hand inequality in B4(b), were standard problems on AP Calculus BC exams.

B5. In class we proved that if $I \subset \mathbf{R}$ is an open interval, $f : I \to \mathbf{R}$ is differentiable, and $f'(x) > 0$ for all $x \in I$, then f is strictly increasing (i.e. $x_1 < x_2$ implies $f(x_1) < f(x_2)$). In this problem we show that the requirement "$f'(x) > 0$ for <u>all</u> $x \in I$" can be somewhat relaxed without affecting the conclusion. Parts (a), (b), (c), and (e) draw successively stronger conclusions, by using successively weaker hypotheses. Each problem-part is intended to help you to the next part, with the exception that part (d) is independent of parts (a), (b), and (c).

(a) Let $a, b \in \mathbf{R}$ be given, with $a < b$. Assume that $f : [a, b] \to \mathbf{R}$ is continuous, is differentiable on the open interval (a, b), and that $f'(x) > 0$ for all $x \in (a, b)$. Prove that f is strictly increasing on the closed interval $[a, b]$.

(b) Let $a, b \in \mathbf{R}$ be given, with $a < b$. Assume that $f : [a, b] \to \mathbf{R}$ is continuous, is differentiable on the open interval (a, b), that $f'(x) \geq 0$ for all $x \in (a, b)$, and that $f'(x) = 0$ for at most finitely many $x \in (a, b)$. Prove that f is strictly increasing on the closed interval $[a, b]$.

(c) Let $a, b \in \mathbf{R}$ be given, with $a < b$. Assume that $f : (a, b) \to \mathbf{R}$ is differentiable and that $f'(x) \geq 0$ for all $x \in (a, b)$. Let $Z(f') = \{x \in (a, b) \mid f'(x) = 0\}$ (the *zero-set* of f'), and assume that $Z(f')$ has no cluster points in the open interval (a, b). Prove that f is strictly increasing on (a, b).

(d) Let $a, b \in \mathbf{R}$ be given, with $a < b$. Assume that $f : [a, b] \to \mathbf{R}$ is continuous, and is strictly increasing on the open interval (a, b). Prove that f is strictly increasing on the closed interval $[a, b]$. (Note that no differentiability is assumed; this problem-part is independent of the previous parts, and is intended as a lemma to help you get from part (c) of this problem to part (e).)

(e) Hypotheses as in part (c). Prove that f is strictly increasing on the closed interval $[a, b]$.

(f) Define $f : \mathbf{R} \to \mathbf{R}$ by $f(x) = x - \sin x$. Prove that f is strictly increasing.

B6. Let X and Y be metric spaces, and let $p_0 \in X$. A function $f : X \to Y$ is called *Lipschitz at* p_0, or *Lipschitz continuous at* p_0, if there exist $K, \delta > 0$ such that

$$d_Y(f(p), f(p_0)) \leq K d_X(p, p_0) \tag{1}$$

for all $p \in B_\delta(p_0)$. We call f *Lipschitz* (or *Lipschitz continuous*)—with no "at p_0"—if there exists $K > 0$ such that

$$d_Y(f(p), f(q)) \leq K d_X(p, q) \tag{2}$$

for all $p, q \in X$. We call f *locally Lipschitz* if for all $p_0 \in X$, there exists $\delta > 0$ such that the restriction of f to $B_\delta(p_0)$ is Lipschitz.[1] As you will see below in part (a), "Lipschitz at p_0" implies "continuous at p_0," justifying the terminology "Lipschitz continuous". Note that continuity is a *local* concept; continuity of a function f at a point p_0 involves only the values of f at points "close" to p_0. If X is unbounded, the Lipschitz condition (2) imposes strong conditions not just on how rapidly $d_Y(f(p), f(q))$ gets *small* as $q \to p$, but on how rapidly $d_Y(f(p), f(q))$ can grow as q gets very far from q. Thus, for continuity issues, "locally Lipschitz" is a more relevant concept than just-plain Lipschitz.

(a) Prove that if $f : X \to Y$ is Lipschitz at $p_0 \in X$, then f is continuous at p_0.
(*Note*: The converse is false. For example, the square-root function from $[0, \infty)$ to $\mathbf{R}$ is continuous but is not Lipschitz at 0.)

For the remainder of this problem, let $I \subset \mathbf{R}$ be an open interval, and $f : I \to \mathbf{R}$ a function.

(b) Let $x_0 \in I$ be given. Prove that if f is differentiable at x_0, then f is Lipschitz at x_0.

(c) Prove that if f is differentiable, and the function $f' : I \to \mathbf{R}$ is bounded, then f is Lipschitz.

(d) Prove that if f is differentiable, and the function $f' : I \to \mathbf{R}$ is continuous, then f is locally Lipschitz.

[1] Note that "locally Lipschitz" is stronger than "Lipschitz continuous at every point"; for the latter, there would be a $K(q)$ that works in (2) for each $q \in X$ and all p sufficiently close to q, but there might not be a single K that works simultaneously for all p, q sufficiently close to a given p_0. Somewhat more logical terminology for "locally Lipschitz", which Dr. Groisser prefers, is "locally uniformly Lipschitz", which appears in some textbooks—e.g. Loomis & Sternberg, *Advanced Calculus*, Addison-Wesley (1968), p. 267—but most mathematicians do not insert "uniformly" into this terminology. Similarly, somewhat more logical terminology than "Lipschitz continuous" (with no "at p_0) might be "uniformly Lipschitz", but this is not standard terminology.