MAA 4212, Spring 2019—Assignment 1's non-book problems

B1. Let X and Y be metric spaces, and let $p \in X$. A function $f : X \to Y$ is called *Lipschitz at* p, or *Lipschitz continuous at* p, if there exist $K > 0$ and $\delta > 0$ such that for all $q \in B_\delta(p) := B_\delta^X(p)$,

$$d_Y(f(q), f(p)) \leq K d_X(q, p). \tag{1}$$

(Note that, given X, Y, and f, if there is some $K \in \mathbf{R}$ that "works" in (1), then any larger K also works, and hence there is some positive K that works. Hence the "$K > 0$" requirement in the definition is superfluous, but is convenient for situations in which we might want to divide by K.)

We call f *Lipschitz* (or *Lipschitz continuous*)—with no "at p_0"—if there exists $K > 0$ such that for all $p, q \in X$, inequality (1) holds. (See **Discussion of the terminology "Lipschitz function"** at the end of this assignment.)

(a) Prove that if $f : X \to Y$ is Lipschitz at $p_0 \in X$, then f is continuous at p_0 (justifying the terminology "Lipschitz continuous").

(*Note*: The converse is false. For example, the square-root function from $[0, \infty)$ to $\mathbf{R}$ is continuous but is not Lipschitz at 0.)

(b) Prove that if $f : X \to Y$ is Lipschitz then f is *uniformly* continuous.

(Again, the converse is false, with the square-root function providing a counterexample.)

B2. Let X be a nonempty set, let (Y, d_Y) be a metric space, and let $B(X, Y)$ denote the set of all bounded functions from X to Y. (Note that if Y is a bounded metric space, then *all* functions $X \to Y$ are bounded, so in this case $B(X, Y)$ is the set of *all* functions $X \to Y$.) Let D' be the uniform metric on $B(X, Y)$, as defined in class:

$$D'(f, g) = \sup\{d_Y(f(x), g(x)) \mid x \in X\}$$

Below, "$B(X, Y)$" is used as short-hand for the metric space $(B(X, Y), D')$.

In class we proved most of the following:

$$\begin{array}{l}\text{A sequence } (f_n)_{n=1}^{\infty} \text{ in } B(X,Y) \text{ converges in } B(X,Y) \\ \text{if and only if } (f_n) \text{ converges uniformly.}\end{array} \tag{2}$$

Omitted from classwork was the proof that if a sequence (f_n) converges uniformly, then the (pointwise) limit function is bounded. This ingredient is the result of Rosenlicht problem IV.38, provided you assume in #38 that X is merely a nonempty set, rather than a metric space. Below, you may assume fact (2).

(a) Prove that if Y is complete, then $B(X, Y)$ is complete.

(b) Show that $B(\mathbf{N}, \mathbf{R}) = \ell^\infty(\mathbf{R})$ (i.e. the underlying sets are the same, and the metrics are the same). Thus, part (a) above shows that $\ell^\infty(\mathbf{R})$ is complete—a fact you proved a different way if you were in my section last fall and succeeded in doing problem B4(c) on Homework Assignment 5, or were in a different section and succeeded in doing the "challenge" part of Problem 6.7 in your section's lecture notes. (My "$\ell^\infty(\mathbf{R})$" is the "ℓ^∞" in those notes' Problem 6.7.)

(c) For $n \in \mathbf{N}$, let $J_n = \{1, 2, \ldots, n\}$. Note that, for any n, *every* function $J_n \to \mathbf{R}$ is bounded. What is the relation between the metric spaces $B(J_n, \mathbf{R})$ and $(\mathbf{R}^n, d_\infty)$?

B3. Now let both X and Y be nonempty metric spaces, and let $BC(X,Y) \subset B(X,Y)$ denote the set of bounded *continuous* functions $X \to Y$. Let D denote the restriction to $BC(X,Y) \times BC(X,Y)$ of the uniform metric D on $B(X,Y)$. (As noted in class, if X is compact then every continuous function $X \to Y$ is bounded, so that $BC(X,Y) = C(X,Y)$ in this case, and D is exactly the uniform metric on $C(X,Y)$ in this case.)

(a) Prove that $BC(X,Y)$ is a closed subset of $(B(X,Y), D')$.

(b) Prove that if Y is complete, then $(BC(X,Y), D)$ is complete. (In class we proved that if X is compact and Y is complete, then $C(X,Y)$ is complete. What you're proving here is a generalization that does not require assuming X to be compact.)

Remark. In many areas of mathematics, an important question is: given a sequence of functions that have some nice properties, if that sequence converges (in whatever sense is of interest in context), does the limit function have those same nice properties? Part of what fact (2) in problem B2, and problem B3, are asserting is that for the "nice properties" of boundedness (in the setting of B2), or boundedness and continuity (in the setting of B3), the answer is "yes" if our notion of convergence is uniform convergence or, equivalently (by fact (2)), convergence in the metric space $B(X,Y)$.

Discussion of the terminology "Lipschitz function". The definition of "Lipschitz function" can be rewritten as: f is Lipschitz if

$$\exists K > 0 \text{ such that } \forall p \in X \text{ and } \forall q \in X, \text{ inequality (1) holds.}$$

This notion can be generalized several ways. Temporarily (and arbitrarily) numbering some properties that every Lipschitz function has, let's say that a general function $f : X \to Y$ has:

- Property 1a if

$$\forall p \in X, \ \exists K > 0 \text{ such that } \forall q \in X, \text{ inequality (1) holds.}$$

- Property 1b if

$$\forall p \in X \text{ and } \forall \delta > 0, \ \exists K > 0 \text{ such that } \forall q \in B_\delta(p), \text{ inequality (1) holds.}$$

- Property 1c if

$$\forall p \in X, \exists \delta > 0 \text{ and } \exists K > 0 \text{ such that } \forall q \in B_\delta(p), \text{ inequality (1) holds.}$$

- Property 2 if for all $p_0 \in X$, there exists $\delta_0 > 0$ such that the restriction of f to $B_{\delta_0}(p_0)$ is Lipschitz; equivalently, if

$$\forall p_0 \in X,\ \exists \delta_0 > 0 \text{ and } K > 0 \text{ such that } \forall p, q \in B_{\delta_0}(p_0),\ \text{inequality (1) holds.}$$

(We could also define f to have Property 2a, 2b, or 2c if for all $p_0 \in X$, there exists $\delta_0 > 0$ such that the restriction of f to $B_{\delta_0}(p_0)$ has Property 1a, 1b, or 1c respectively, but the properties obtained this way are not very interesting or useful, and if you write them out with quantifiers they'll make your head hurt.)

Observe that in Property 1a, K can depend on p; in Property 1b, K can depend on p and δ; in Property 1c, both δ and K can depend on p, and each of δ and K can depend on the other. Similarly, in Property 2, both δ_0 and K can depend on p, and each of δ_0 and K can depend on the other. But in our definition of "Lipschitz function", K does not depend on any choice of point, and there is no visible δ for it to depend on. Thus, our definition of "f is Lipschitz" involves the principle of *uniformity* in two ways: the "K" that can depend on various parameters in our four generalized properties, doesn't, and the δ or δ_0 that can depend on other parameters in the generalized properties 1a, 1b, 1c, and 2, is vacuously independent of parameters (in the definition of "f is Lipschitz") by virtue of being absent.

Property 2 is a property that will arise naturally once we learn about differentiation, and has an actual name: f is called *locally Lipschitz* if it has Property 2.

Next, looking back at the definition of "f is Lipschitz at p", observe that Property 1c is equivalent to: f is Lipschitz at p for every $p \in X$. Observe that this is *weaker* than "f is locally Lipschitz" (Property 2 implies Property 1c—if we look at the special case $q = p_0$ in Property 2, we get Property 1c—but not vice-versa).

Based on common conventions in mathematics, "function with Property 1c"—i.e. "function that is Lipschitz at every point"—*ought* to be the definition of "Lipschitz function" (think about how "continuous function" was defined), but unfortunately it isn't; in fact there is no standard, short name for Property 1c. This isn't a great inconvenience, because Property 1c rarely appears in theorems: it turns out that for any property relating to "Lipschitz" to be useful as a hypothesis (other than to deduce continuity), some uniformity of the K in inequality (1) is needed, and it turns out in most circumstances in which we're able to prove that a function f is Lipschitz at each point, we can prove the stronger statement that f is locally Lipschitz.

Nonetheless, a more logical name for what we are calling "Lipschitz function" is "*uniformly* Lipschitz function", and a more logical name for what we are calling "locally Lipschitz function" is "locally *uniformly* Lipschitz function". Some mathematicians, myself included (outside of this class), *do* include "uniformly" in these terms (see e.g. a textbook that I learned from: Loomis & Sternberg, *Advanced Calculus*, Addison-Wesley

(1968), p. 267). If I were king, everyone would include the "uniformly", but since my chances of becoming king are slim, you should learn what *most* people mean by "Lipschitz function" and "locally Lipschitz function", not just what I wish the terminology were.

As you saw in problem B1, Lipschitz functions are continuous. Note that continuity is a *local* concept; continuity of a function f at a point p_0 involves only the values of f at points "close" to p_0. If the metric space X is unbounded, the Lipschitz condition (1) for a function $f : X \to Y$ imposes strong conditions not just on how rapidly $d_Y(f(p), f(q))$ gets *small* as $q \to p$, but on how rapidly $d_Y(f(p), f(q))$ can *grow* as q gets very *far* from p. Thus, for continuity issues, "locally Lipschitz" is a more relevant concept than just-plain Lipschitz.