MAA 4212, Spring 2019—Assignment 2's non-book problems

B1. Let X and Y be metric spaces, $(f_n : X \to Y)_{n=1}^{\infty}$ a sequence of functions, and $f : X \to Y$ a function. Assume that there is a real-valued sequence $(c(n))_{n=1}^{\infty}$ such that (i) for all $n \in \mathbf{N}$ and $x \in X$, $d_Y(f_n(x), f(x)) \leq c(n)$, and (ii) $\lim_{n\to\infty} c(n) = 0$. Prove that (f_n) converges uniformly to f.

Thus, to prove that a sequence (f_n) converges uniformly to a given function f, it suffices to find, for each n, a uniform upper bound $c(n)$ on the distances $d_Y(f_n(x), f(x))$ (where "uniform" means "independent of x"), with the property that $c(n) \to 0$ as $n \to \infty$. In practice, this is virtually always how uniform convergence is shown (for a sequence of functions that *does* converge uniformly). For example, on the previous assignment you probably did Rosenlicht problem IV.34(a) this way.

Exercises on the Mean Value Theorem

All of the exercises below (though not all *parts* of them) make use of the Mean Value Theorem (MVT) or its corollaries, in one form or another, but some require you to use other theorems in addition. You may assume that the trigonometric and inverse trigonometric functions have the derivatives you learned in Calculus I-II-III.

B2. Let $I \subset \mathbf{R}$ be an open interval, and $f : I \to \mathbf{R}$ a function.

(a) Let $x_0 \in I$. Prove that if f is differentiable at x_0, then f is Lipschitz at x_0.

(b) Prove that if f is differentiable, and the function $f' : I \to \mathbf{R}$ is bounded, then f is Lipschitz.

(c) Prove that if f is differentiable, and the function $f' : I \to \mathbf{R}$ is continuous, then f is locally Lipschitz. (Recall from the terminology discussion in Assignment 1's non-book problems that a function $g : X \to Y$, where X and Y are metric spaces, is *locally* Lipschitz if for all $p \in X$, there exists $\delta > 0$ such that the restriction of g to $B_\delta(p)$ is Lipschitz.)

B3. Let $a, b \in \mathbf{R}, a < b$.

(a) Assume that $f, g : [a, b) \to \mathbf{R}$ are continuous, and are differentiable on (a, b). Assume also that $f(a) = g(a)$ and that $f'(x) > g'(x)$ for all $x \in (a, b)$. Prove that $f(x) > g(x)$ for all $x \in (a, b)$.

(b) Assume that $f, g : (a, b] \to \mathbf{R}$ are continuous, and are differentiable on (a, b). Assume also that $f(b) = g(b)$ and that $f'(x) > g'(x)$ for all $x \in (a, b)$. In this case, what order-relation do $f(x)$ and $g(x)$ obey for $x \in (a, b)$?

In part (b), you are not being asked for a formal proof. Just state how, under the hypotheses in (b), any inequalities you used for the proof in part (a) become modified, and how this affects or does not affect the conclusion.

B4. Prove that

$$\text{(a)} \qquad \frac{x}{1+x^2} < \tan^{-1} x < x \quad \text{for all } x > 0,$$

and

$$\text{(b)} \qquad x < \sin^{-1} x < \frac{x}{\sqrt{1-x^2}} \quad \text{for } \ 0 < x < 1.$$

(Here "$\tan^{-1}$" and "$\sin^{-1}$" are the inverse tangent and inverse sine functions, also known as "arctan" and "arcsin" respectively.)

B5. Prove that, for all $x > 0$,

$$\text{(a)} \quad \sin x < x,$$

$$\text{(b)} \quad \cos x > 1 - \frac{x^2}{2}, \qquad \text{and}$$

$$\text{(c)} \quad x - \frac{x^3}{3!} < \sin x < x - \frac{x^3}{3!} + \frac{x^5}{5!}\ .$$

(Warning: if you try to use Taylor's Theorem for this problem—which is not recommended—don't forget that numbers of the form "$\sin c$" or "$\cos c$" can be negative as well as positive!)

B6. In class we proved that if $I \subset \mathbf{R}$ is an interval, $f : I \to \mathbf{R}$ is continuous on I and differentiable on I°, and $f'(x) > 0$ for all $x \in I^\circ$, then f is strictly increasing (i.e. $x_1 < x_2$ implies $f(x_1) < f(x_2)$). In this problem we show that the requirement "$f'(x) > 0$ for <u>all</u> $x \in I^\circ$" can be somewhat relaxed without affecting the conclusion. Parts (a), (b), and (d) draw successively stronger conclusions, by using successively weaker hypotheses. Each problem-part is intended to help you do the next part, with the exception that part (c) is independent of parts (a) and (b).

(a) Let $a, b \in \mathbf{R}$, with $a < b$. Assume that $f : [a, b] \to \mathbf{R}$ is continuous, is differentiable on the open interval (a, b), that $f'(x) \geq 0$ for all $x \in (a, b)$, and that $f'(x) = 0$ for at most finitely many $x \in (a, b)$. Prove that f is strictly increasing on the closed interval $[a, b]$.

(b) Let $a, b \in \mathbf{R}$, with $a < b$. Assume that $f : (a, b) \to \mathbf{R}$ is differentiable and that $f'(x) \geq 0$ for all $x \in (a, b)$. Let $Z(f') = \{x \in (a, b) \mid f'(x) = 0\}$ (the *zero-set* of f'), and assume that $Z(f')$ has no cluster points in the open interval (a, b). Prove that f is strictly increasing on (a, b).

(c) Let $a, b \in \mathbf{R}$, with $a < b$. Assume that $f : [a, b] \to \mathbf{R}$ is continuous, and is strictly increasing on the open interval (a, b). Prove that f is strictly increasing on the closed interval $[a, b]$. (Note that no differentiability is assumed; this problem-part is independent of the previous parts, and is intended as a lemma to help you get from part (b) of this problem to part (d).)

(d) Hypotheses as in part (b). Prove that f is strictly increasing on the closed interval $[a, b]$.

(e) Define $f : \mathbf{R} \to \mathbf{R}$ by $f(x) = x - \sin x$. Prove that f is strictly increasing.