MAA 4212, Spring 2019—Assignment 3's non-book, non-notes problem

B1. Let $I \subset \mathbf{R}$ be an open interval, let $x_0 \in I$, and $f : I \to \mathbf{R}$ be a function that is continuous on I and differentiable on $I \setminus \{x_0\}$. Assume that $\lim_{x \to x_0^+} f'(x)$ and $\lim_{x \to x_0^-} f'(x)$ exist and are equal. Prove that f is differentiable at x_0 and that $f'(x_0)$ has the same value as these two limits (and hence that f' not only exists at x_0 but is continuous there).

Be careful not to assume that f has any properties not given in the hypotheses. For example, don't assume that f' is continuous on $I \setminus \{x_0\}$.