MAA 4212, Spring 2019—Assignment 7's non-book problems

B1. This problem, a slightly rewritten version of one that was originally going to be on your midterm, is a variation on the first page and half (roughly) of Rosenlicht Section VII.4. It defines certain functions via power series, and shows without any use of Taylor's Theorem that the functions defined this way must be the functions that we've always called sine and cosine.

Consider the following two series:

$$\begin{aligned} \text{Series}_1 = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n} &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots, \\ \text{Series}_2 = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(2n-1)!} x^{2n-1} &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots . \end{aligned}$$

Each of these series can be written in the form $\sum_{n=0}^{\infty} c_n x^n$; we simply have $c_n = 0$ for all odd n in Series_1, and $c_n = 0$ for all even n in Series_2. Thus, both series are power series in x, and we can apply the results we've proven about power series.

(a) Show that Series_1 converges for all $x \in \mathbf{R}$. (Series_2 also converges for all x, but the argument is essentially the same as for Series_1, so I'm not asking you to give both arguments.)

For the remainder of this problem, you may assume that both Series_1 and Series_2 converge for all $x \in \mathbf{R}$. Let $f, g : \mathbf{R} \to \mathbf{R}$ be the functions to which $\text{Series}_1, \text{Series}_2$, respectively, converge pointwise.

(b) State a general result about power series that justifies the following statement:

> The functions f and g are differentiable (on the whole real line), and their derivatives satisfy
>
> $$f'(x) = \sum_{n=1}^{\infty} \frac{d}{dx}\left((-1)^n \frac{x^{2n}}{(2n)!}\right) \quad \text{and} \quad g'(x) = \sum_{n=1}^{\infty} \frac{d}{dx}\left((-1)^{n-1} \frac{x^{2n-1}}{(2n-1)!}\right).$$

(c) Show that $f' = -g$ and $g' = f$, hence also that $f'' = -f$ and $g'' = -g$.

(d) Show that the function $f^2 + g^2$ is constant. (*Hint*: Part (c).) Then use this constancy to show that $f^2 + g^2 \equiv 1$.

(e) Let $h : \mathbf{R} \to \mathbf{R}$ be a twice-differentiable function satisfying $h'' = -h$. Show that the functions $fh - gh'$ and $gh + fh'$ are constant. Then use this to show that

$$\begin{aligned} fh - gh' &\equiv h(0) \qquad \text{and} \\ gh + fh' &\equiv h'(0). \end{aligned}$$

Then use part (d) to show that the above pair of equations implies $h = h(0)f + h'(0)g$.

(f) Deduce from the preceding parts of this problem that if $h : \mathbf{R} \to \mathbf{R}$ satisfies $h'' = -h$, then the values $h(0)$ and $h'(0)$ determine h uniquely, and there is a power series in x converging to $h(x)$ on the whole real line. Furthermore, the unique such function satisfying $h(0) = 1$ and $h'(0) = 0$ is f and has the power-series represention Series_1, and the unique such function satisying $h(0) = 0$ and $h'(0) = 1$ is g and has the power-series representation Series_2.

Epilog: Since these functions are unique, we may give them names: f = cosine, g = sine. The functions that we've always called cosine and sine satisfy the defining conditions in part (f), so they must be identical to the newly defined cosine and sine functions.

B2. Let $M_{n\times n}$ be the space of real $n \times n$ matrices, equipped with the operator norm. (Any norm would lead to the same answers in this problem, since all norms on a given finite-dimensional vector space are equivalent.) Define $F : M_{n\times n} \to M_{n\times n}$ by $F(A) = A^2$.

(a) For all $A, B \in M_{n\times n}$, compute the directional derivative $(D_A F)(B)$. (Remember that matrix multiplication is not commutative!)

(b) Show that F is differentiable. For each $A \in M_{n\times n}$, what is the linear map $DF|_A$?

(c) For the case $n = 1$, the space $M_{n\times n}$ is simply $\mathbf{R}$, so for each $a \in \mathbf{R}$ the Jacobian $J_F(a)$ is the unique number c such that $DF|_a$ is the map $x \mapsto cx$. *Based on your answer to part (b)*, what is the number $J_F(a)$?

(d) For general reasons discussed in class, the number $J_F(a)$ must be the "Calc-1 derivative" $f'(a)$. Check that it is. (If it isn't, you did something wrong earlier.)

(e) The map $\phi : M_{2\times 2} \to \mathbf{R}^4$, $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto (a, b, c, d)^t$, is an isomorphism, so the "equivalent" map $\tilde{F} := \phi \circ F \circ \phi^{-1} : \mathbf{R}^4 \to \mathbf{R}^4$ is differentiable. Note that its Jacobian, at a given point, is a $\underline{4 \times 4}$ matrix, not a 2×2 matrix . For a given, arbitrary, $A \in M_{2\times 2}$, which do you find easier: (i) Writing down the linear map $DF|_A$ the way you did in part (b), or (ii) computing the Jacobian $J_{\tilde{F}}(\phi(A))$ so that you can express the map $D\tilde{F}|_{\phi(A)}$ as multiplication by $J_{\tilde{F}}(\phi(A))$?

(If you answered (ii), would your answer change if the corresponding question were asked for $M_{3\times 3}$? $M_{10\times 10}$? This example is intended to show you that while Jacobians are extremely important and useful, they aren't *always* the most illuminating way to understand derivatives of maps between finite-dimensional vector spaces.)

B3. Let $(V, \| \ \|_V), (W, \| \ \|_W)$ be finite-dimensional normed vector spaces. Given a differentiable function $g : I \to V$, where $I \subset \mathbf{R}$ is an open interval, for each $t \in I$ we define

$$g'(t) = \lim_{h\to 0} \frac{g(t+h)-g(t)}{h} \ ;$$

equivalently $g'(t) = Dg|_t(1) = (D_t g)(1)$.[1] As noted in class, if $V = \mathbf{R}^m$ then the vector $g'(t) \in V$ is simply the $m \times 1$ Jacobian matrix $J_g(t)$.

The Fundamental Theorem of Calculus for real-valued functions generalizes to V-valued functions: if g is as above, and g' is continuous, then for all $a, b \in I$ we have

$$\int_a^b g'(t)\, dt = g(b) - g(a) \ ;$$

you may assume this fact (which is not difficult to prove) below.

Let $U \subset V$ be open and let $F : U \to W$ be a differentiable function.

(a) Let $I \subset \mathbf{R}$ be an open interval, and let $g : I \to U$ be a differentiable function. Show that

$$(F \circ g)'(t) = DF|_{g(t)}(g'(t)).$$

(b) Recall that for $p, q \in V$, the *line segment from p to q* is the set of points $\{(1-t)p + tq \mid 0 \le t \le 1\}$; equivalently, the image of the map $g : [0,1] \to V$ defined by $g(t) = (1-t)p + tq$. A subset $C \subset V$ is called *convex* if for all $p, q \in C$, the line segment from p to q is contained in C. Note that if C is open as well as convex, and $p, q \in C$, then the point $g(t) = (1-t)p + tq$ lies in C for all t in some open interval I containing $[0,1]$, so we may define $g'(0)$ and $g'(1)$ by extending g to such an interval.[2]

Suppose that U is convex. Show that for all $p, q \in U$,

$$F(q) - F(p) = \int_0^1 DF|_{g(t)}(v)\ dt, \tag{1}$$

where $g(t) = (1-t)p + tq$ and $v = q - p$.

(c) Suppose that U is convex and that $\{\| DF|_p \|_{\mathrm{op}} : p \in U\}$ has an upper bound K. (Here the operator norm is the one determined by the given norms on V and W. Show that for all $p, q \in U$ we have

$$\|F(q) - F(p)\|_W \le K\|q - p\|_V \ ; \tag{2}$$

equivalently, $d_W(F(q), F(p)) \le K d_V(q, p)$. Thus K is a Lipschitz constant for F, and F is Lipschitz.

[1] The meaning of the last two expressions is the same as in "$Dg|_p(v)$" and "$(D_p g)(v)$"; in our current setting, the domain of g is a subset of $\mathbf{R} = \mathbf{R}^1$, so the point p is a real number we're choosing to denote t, and the vector v is the real number 1 that h is invisibly multiplying in "$g(t+h)$". As discussed in class, for maps between subsets of finite-dimensional normed vector spaces, the existence and values of limits and derivatives are independent of which norms are used.)

[2] We could also define $g'(0)$ and $g'(1)$ by using one-sided limits, but in order to apply part (a), which we'll be doing later in this problem, we need g to be defined on an open set.

Note that for the case $W = V = \mathbf{R}$, inequality (2) follows from the Mean Value Theorem. However, the vector-valued analog of the Mean Value Theorem would be the statement that the integral on the right-hand side of (1) can be replaced by $DF|_{g(t_0)}(v)$ for some $t_0 \in (0, 1)$. This analog is false; it is easy to find counterexamples even for maps from an interval into $\mathbf{R}^2$.

(d) Show that every open ball in $(V, \| \ \|_V)$ is convex. (This is true of any ball in this space, whether open or not, but you're just being asked to show it for open balls.)

(e) For any metric space (E, d) a map $G : E \to E$ is called a *contracting mapping* (synonyms: *contraction* and *contraction mapping*) if G has a Lipschitz constant $K' < 1$:

$$d(G(x), G(y)) \leq K' \, d(x, y) \quad \text{for all } x, y \in E. \tag{3}$$

Note that this is a more restrictive condition than "$d(G(x), G(y)) < d(x, y) \quad \text{for all } x, y \in E$" (why?).

The *Contracting Mapping Fixed-Point Theorem* (CMFPT) asserts that if $G : E \to E$ is a contraction and (E, d) is complete, then G has a unique fixed point (a point $p \in E$ for which $G(p) = p$). You may assume this theorem below. For this theorem to apply, it is critical that G be defined on *all* of E; it is not enough to have a map $G : (U' \subsetneq E) \to E$ satisfying (3) for some $K' < 1$. However, if $G : (U' \subset E) \to E$ satisfies (3) for some $K' < 1$, where U' is closed and (E, d) is complete, *and* G preserves U' (i.e. $G(U') \subset U'$), then G is a contraction on the complete metric space (U', d).

Show that if (i) U' is an open ball $B_r(x_0)$ in a complete metric space (E, d), and (ii) $G : U' \to E$ satisfies (3) for some $K' < 1$, and (iii) $d(G(x_0), x_0) < (1 - K')r$, then G preserves the closed ball $U'' = \bar{B}_{r'}(x_0)$ for some $r' < r$. Hence if these conditions are met, the CMFPT implies that $G|_{U''}$ has a unique fixed point. In fact, G itself has a unique fixed point in the open ball $B_r(x_0)$, since if x_0 is the fixed-point of $G|_{U''}$, and x_1 is a different fixed-point of G, then (3) implies the contradiction $d(x_1, x_0) = d(G(x_1), G(x_0)) \leq K' d(x_1, x_0) < d(x_1, x_0)$.

(f) Using the earlier parts of this problem, show the following: if (i) U is an open ball $B_r(p_0)$, and (ii) there exists $K < 1$ such that for all $p \in U$ we have $\| DF|_p \|_{\text{op}} \leq K$, and (iii) $d_V(F(p_0), p_0) < (1 - K)r$, then F has a unique fixed-point.