MAA 4212, Spring 2020—Assignment 1's non-book problems

B1. Let X and Y be metric spaces, and let $p \in X$. A function $f : X \to Y$ is called *Lipschitz at p*, or *Lipschitz continuous at p*, if there exist $K > 0$ and $\delta > 0$ such that for all $q \in B_\delta(p) := B^X_\delta(p)$,

$$d_Y(f(q), f(p)) \le K d_X(q, p). \tag{1}$$

(Note that, given X, Y, and f, if there is some $K \in \mathbf{R}$ that "works" in (1), then any larger K also works, and hence there is some positive K that works. Hence the "$K > 0$" requirement in the definition is superfluous, but is convenient for situations in which we might want to divide by K.)

We call f *Lipschitz* (or *Lipschitz continuous*)—with no "at p_0"—if there exists $K > 0$ such that for all $p, q \in X$, inequality (1) holds. (See **Discussion of the terminology "Lipschitz function"** at the end of this assignment.)

(a) Prove that if $f : X \to Y$ is Lipschitz at $p_0 \in X$, then f is continuous at p_0 (justifying the terminology "Lipschitz continuous").

(*Note*: The converse is false. For example, the square-root function from $[0, \infty)$ to $\mathbf{R}$ is continuous but is not Lipschitz at 0.)

(b) Prove that if $f : X \to Y$ is Lipschitz then f is *uniformly* continuous.

(Again, the converse is false, with the square-root function providing a counterexample.)

In the fall, we discussed several properties that a given metric space may or may not have. Among these were boundedness, completeness, and compactness. (Recall that, by definition, a metric space is bounded if it is a bounded subset of itself.) We can ask whether function-spaces $B(X, Y)$, $BC(X, Y)$ (defined in problem B3), and $C(X, Y)$ (when X is compact), have these properties when X and/or Y have them. The remaining problems (B2 and B3) answer these questions.

B2. Let X be a nonempty set and let (Y, d_Y) be a nonempty metric space. As in class, let $B(X, Y)$ denote the set of all bounded functions from X to Y. Let D' be the uniform metric on $B(X, Y)$, as defined in class: $D'(f, g) = \sup\{d_Y(f(x), g(x)) \mid x \in X\}$. Below, "$B(X, Y)$" is used as short-hand for the metric space $(B(X, Y), D')$.

(a) Prove that $B(X, Y)$ is bounded if and only if Y is bounded.

Hint for the "only if" part: To show that if Y is not bounded then neither is $B(X, Y)$, consider sequences of *constant* functions—i.e. sequences $(f_n : X \to Y)_{n=1}^\infty$ for which each term f_n is a constant function.[1]

[1] This is a good reminder of the importance of word-order: A *sequence of constant functions* $X \to Y$ is not the same thing as a *constant sequence of functions* $X \to Y$! The latter means a sequence $(f_n : X \to Y)_{n=1}^\infty$ for which $f_1 = f_2 = f_3 = \dots$; the common "value" of the sequence (the function f_1) may or may not be a constant function.

Remark: If Y is a bounded metric space, then *all* functions $X \to Y$ are bounded, so in this case $B(X,Y)$ is the set of *all* functions $X \to Y$.

(b) Prove $B(X,Y)$ is complete if and only if Y is complete.

Hint for the "only if" part: Same as for part (a).

(c) Show that $B(\mathbf{N},\mathbf{R}) = \ell^\infty(\mathbf{R})$ (i.e. the underlying sets are the same, and the metrics are the same). Thus, part (a) above shows that $\ell^\infty(\mathbf{R})$ is complete—a fact you proved a different way last fall if you succeeded in doing problem B4(c) on Homework Assignment 5.

(d) For $n \in \mathbf{N}$, let $J_n = \{1, 2, \ldots, n\}$. Note that, for any n, *every* function $J_n \to \mathbf{R}$ is bounded. What is the relation between the metric spaces $B(J_n, \mathbf{R})$ and $(\mathbf{R}^n, d_\infty)$?

B3. Now let both X and Y be nonempty metric spaces, and let $BC(X,Y) \subset B(X,Y)$ denote the set of bounded *continuous* functions $X \to Y$. Let D denote the restriction to $BC(X,Y)$ of the uniform metric D' on $B(X,Y)$. Below, "$BC(X,Y)$" is used as shorthand for the metric space $(BC(X,Y), D)$.

(a) Prove that $BC(X,Y)$ is a closed subset of $(B(X,Y), D')$.

(b) Prove that $BC(X,Y)$ is bounded if and only if Y is bounded.

(c) Prove that $BC(X,Y)$ is complete if and only if Y is complete.

(d) As noted in class, if X is compact then every continuous function $X \to Y$ is bounded, so that $BC(X,Y) = C(X,Y)$. Thus, as corollaries of (a), (b), and (c), deduce that when X is compact, the corresponding statements are true with $BC(X,Y)$ replaced by $C(X,Y)$.

(e) Show that even if both X and Y are compact, the metric space $C(X,Y)$ need not be compact.

Hint: To show this, we need only produce *one* compact metric space X and *one* compact metric space Y for which $C(X,Y)$ is not compact. Since the subspace $[0,1] \subset \mathbf{R}$ is compact, we'll be done if we can show that $C([0,1],[0,1])$ is not compact; equivalently, if $C([0,1],[0,1])$ is not *sequentially* compact. Among the examples of sequences of functions $(f_n : [0,1] \to \mathbf{R})_{n=1}^\infty$ that we looked at the first week of class, there's an example in which all the functions were continuous and had range in $[0,1]$ (so we could have replaced the codomain by $[0,1]$), but for which you should be able to show (indirectly) that there is no D-convergent subsequence.

Discussion of the terminology "Lipschitz function". The definition of "Lipschitz function" can be rewritten as: f is Lipschitz if

$$\exists K > 0 \text{ such that } \forall p \in X \text{ and } \forall q \in X, \text{ inequality (1) holds.}$$

This notion can be generalized several ways. Temporarily (and arbitrarily) numbering some properties that every Lipschitz function has, let's say that a general function $f : X \to Y$ has:

- Property 1a if

$$\forall p \in X,\ \exists K > 0 \text{ such that } \forall q \in X,\ \text{inequality (1) holds.}$$

- Property 1b if

$$\forall p \in X \text{ and } \forall \delta > 0,\ \exists K > 0 \text{ such that } \forall q \in B_\delta(p),\ \text{inequality (1) holds.}$$

- Property 1c if

$$\forall p \in X, \text{there exist } \delta > 0 \text{ and } K > 0 \ \text{ such that } \forall q \in B_\delta(p),\ \text{inequality (1) holds.}$$

- Property 2 if for all $p_0 \in X$, there exists $\delta_0 > 0$ such that the restriction of f to $B_{\delta_0}(p_0)$ is Lipschitz; equivalently, if

$$\forall p_0 \in X,\ \text{there exist } \delta_0 > 0 \text{ and } K > 0 \text{ such that } \forall p, q \in B_{\delta_0}(p_0),\ \text{inequality (1) holds.}$$

(We could also define f to have Property 2a, 2b, or 2c if for all $p_0 \in X$, there exists $\delta_0 > 0$ such that the restriction of f to $B_{\delta_0}(p_0)$ has Property 1a, 1b, or 1c respectively, but the properties obtained this way are not very interesting or useful, and if you write them out with quantifiers they'll make your head hurt.)

Observe that in Property 1a, K can depend on p; in Property 1b, K can depend on p and δ; in Property 1c, both δ and K can depend on p, and each of δ and K can depend on the other. Similarly, in Property 2, both δ_0 and K can depend on p, and each of δ_0 and K can depend on the other. But in our definition of "Lipschitz function", K does not depend on any choice of point, and there is no visible δ for it to depend on. Thus, our definition of "f is Lipschitz" involves the principle of *uniformity* in two ways:

(i) the "K" that can depend on various parameters in our four generalized properties, doesn't, and

(ii) the δ or δ_0 that can depend on other parameters in properties 1a, 1b, 1c, and 2, is vacuously independent of parameters (by virtue of there *being* no other parameters in the definition of "f is Lipschitz").

Property 2 is a property that will arise naturally once we learn about differentiation, and (unlike properties 1a, 1b, and 1c) has an actual name: f is called *locally Lipschitz* if it has Property 2.

Next, looking back at the definition of "f is Lipschitz at p", observe that Property 1c is equivalent to: f is Lipschitz at p for every $p \in X$. Observe that this is *weaker* than

"f is locally Lipschitz" (Property 2 implies Property 1c—if we look at the special case $q = p_0$ in Property 2, we get Property 1c—but not vice-versa).

Based on common conventions in mathematics, "function with Property 1c"—i.e. "function that is Lipschitz at every point"—*ought* to be the definition of "Lipschitz function" (think about how "continuous function" was defined), but unfortunately it isn't; in fact there is no standard, short name for Property 1c. This isn't a great inconvenience, because Property 1c rarely appears in theorems: it turns out that for any property relating to "Lipschitz" to be a useful hypothesis for deducing anything other than plain-old continuity, some uniformity of the K in inequality (1) is needed, and it turns out in most circumstances in which we're able to prove that a function f is Lipschitz at each point, we can prove the stronger statement that f is locally Lipschitz.

Nonetheless, a more logical name for what we are calling "Lipschitz function" is "*uniformly* Lipschitz function", and a more logical name for what we are calling "locally Lipschitz function" is "locally *uniformly* Lipschitz function". Some mathematicians, myself included (outside of this class), *do* include "uniformly" in these terms (see e.g. a textbook that I learned from: Loomis & Sternberg, *Advanced Calculus*, Addison-Wesley (1968), p. 267). If I were king, everyone would include the "uniformly" in these terms, but since my chances of becoming king are slim, you should learn what *most* people mean by "Lipschitz function" and "locally Lipschitz function", not just what I wish the terminology were.

As you saw in problem B1, Lipschitz functions are continuous. Note that continuity is a *local* concept; continuity of a function f at a point p_0 involves only the values of f at points "close" to p_0. If the metric space X is unbounded, the Lipschitz condition (1) for a function $f : X \to Y$ imposes strong conditions not just on how rapidly $d_Y(f(p), f(q))$ gets *small* as $q \to p$, but on how rapidly $d_Y(f(p), f(q))$ can *grow* as q gets very *far* from p. Thus, for continuity issues, "locally Lipschitz" is a more relevant concept than just-plain Lipschitz.