## MAA 4212, Spring 2020—Assignment 2's non-book problems

B1. Let $X$ and $Y$ be metric spaces, $(f_n : X \to Y)_{n=1}^{\infty}$ a sequence of functions, and $f : X \to Y$ a function. Assume that there is a real-valued sequence $(c(n))_{n=1}^{\infty}$ such that (i) for all $n \in \mathbf{N}$ and $x \in X$, $d_Y(f_n(x), f(x)) \le c(n)$, and (ii) $\lim_{n\to\infty} c(n) = 0$. Prove that $(f_n)$ converges uniformly to $f$.

Thus, to prove that a sequence $(f_n)$ converges uniformly to a given function $f$, it suffices to find, for each $n$, a uniform upper bound $c(n)$ on the distances $d_Y(f_n(x), f(x))$ (where "uniform" means "independent of $x$"), with the property that $c(n) \to 0$ as $n \to \infty$. In practice, this is virtually always how uniform convergence is shown (for a sequence of functions that *does* converge uniformly). For example, on the previous assignment you probably did Rosenlicht problem IV.34(a) this way.

B2. Prove the following lemma, which has many uses:

**Lemma 0.1 ("Substitution lemma for limits")** *Let $X, Y, Z$ be metric spaces, let $U \subset X$, let $V \subset Y$, let $x_0 \in X$, let $y_0 \in Y$, and assume that $x_0$ and $y_0$ are cluster points of $U$ and $V$ respectively. Let $f : U \to Y$ and $g : V \to Z$ be functions for which $f(U \setminus \{x_0\}) \subset V$, and assume that $\lim_{y\to y_0} g(y)$ exists and that $\lim_{x\to x_0} f(x) = y_0$. Then*

$$\lim_{x\to x_0} g(f(x)) = \lim_{y\to y_0} g(y). \tag{1}$$

**Some comments about this lemma**:

(1) Note that above, as in any statements involving expressions such as "$\lim_{x\to x_0} f(x)$" and "$\lim_{y\to y_0} g(y)$", $f$ and $g$ do not *need* to be defined at $x_0$ and $y_0$ (respectively), but are *allowed* to be defined there. If $x_0 \in U$ and $f(x_0) \in V$, that's fine and dandy; these are just properties that have no bearing on the limits in the lemma. All that matters about the domains and codomains $f$ and $g$ is that $x_0$ and $y_0$ be cluster points of the corresponding domains (so that the notions of "$\lim_{x\to x_0} f(x)$" and "$\lim_{y\to y_0} g(y)$" are defined) and that $f(U \setminus \{x_0\}) \subset \text{domain}(g)$ (so that $g(f(x))$ is defined for all $x \in U \setminus \{x_0\}$). In fact, the latter condition can be weakened to: $f(B_\delta^U(x_0) \setminus \{x_0\}) \subset \text{domain}(g)$ for $\delta > 0$.

(2) If we were to replace $f$ and $g$ in Lemma 0.1 by functions $f : X \to Y$ and $g : Y \to Z$ that are *continuous* at $x_0$ and $y_0$ respectively, and for which $y_0 = f(x_0)$, then the right-hand side of equation (1) would reduce to $g(f(x_0))$. Equation $\lim_{y\to y_0} g(y)$ would simply be saying that $g \circ f$ is continuous at $x_0$—a fact we've previously proven.

(3) Observe that Lemma 0.1 is similar in spirit to one direction of the "sequential characterization of limits of functions". If we were to replace $f$ by a function from $\mathbf{N}$ to domain$(g)$—i.e. a sequence $(y_n)_{n=1}^{\infty}$ in domain$(g)$—and assume that this sequence converges to $y_0$, the statement analogous to equation (1) would then be $\lim_{n\to\infty} g(y_n) = \lim_{y\to y_0} g(y)$.

(4) **Two simple examples.** Suppose that (i) $U \subset \mathbf{R}$ is an open interval containing a point $x_0$ and we are given a function $g : U \setminus \{x_0\} \to \mathbf{R}$, or (ii) $U \subset \mathbf{R}$ is an open interval containing 0 and we are given a function $g : U \setminus \{0\} \to \mathbf{R}$. It is easy to prove directly from "$\epsilon$-$\delta$" definitions that in instance (i) we have $\lim_{x\to x_0} g(x) = \lim_{h\to 0} g(x_0 + h)$, and that in instance (ii), for any nonzero $c \in \mathbf{R}$ we have $\lim_{x\to 0} g(cx) = \lim_{x\to 0} g(x)$. However, we can also obtain these facts as a corollaries (or special cases) of Lemma 0.1. The only subtlety is that in (i) and (ii) we have not assumed that the limit of $g$ exists (as $x \to x_0$ or as $x \to 0$, accordingly), whereas Lemma 0.1 does have such a hypothesis. But recall that by convention, in the absence of any assumption that either of the relevant limits exists, a statement of the form "limit #1 = limit #2" means that, if either limit exists, then so does the other, in which case the limits are equal. (Equivalently: either both limits exist and are equal, or neither exists.) In instance (ii), if we assume existence of $\lim_{x\to 0} g(x)$ (think "$\lim_{y\to 0} g(y)$" in the notation used in Lemma 0.1) and apply the lemma with $f(x) = cx$, we obtain existence of $\lim_{x\to 0} g(cx)$. If we assume existence of $\lim_{x\to 0} g(cx)$ and apply the lemma with $f(x) = x/c$, we obtain existence of $\lim_{x\to 0} g(x)$. (And, of course, under either assumption we obtain the equality of limits.) Similar considerations apply in instance (i).

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## Exercises on the Mean Value Theorem

All of the exercises below (though not all *parts* of them) make use of the Mean Value Theorem (MVT) or its corollaries, in one form or another, but some require you to use other theorems in addition. You may assume that the trigonometric and inverse trigonometric functions have the derivatives you learned in Calculus I-II-III.

B3. Let $I \subset \mathbf{R}$ be an open interval, and $f : I \to \mathbf{R}$ a function.

(a) Let $x_0 \in I$. Prove that if $f$ is differentiable at $x_0$, then $f$ is Lipschitz at $x_0$.

(b) Prove that if $f$ is differentiable, and the function $f' : I \to \mathbf{R}$ is bounded, then $f$ is Lipschitz.

(c) Prove that if $f$ is differentiable, and the function $f' : I \to \mathbf{R}$ is continuous, then $f$ is locally Lipschitz. (Recall from the terminology discussion in Assignment 1's non-book problems that a function $g : X \to Y$, where $X$ and $Y$ are metric spaces, is *locally* Lipschitz if for all $p \in X$, there exists $\delta > 0$ such that the restriction of $g$ to $B_\delta(p)$ is Lipschitz.)

B4. Let $a, b \in \mathbf{R}$ and assume $a < b$.

(a) Assume that $f, g : [a, b) \to \mathbf{R}$ are continuous, and are differentiable on $(a, b)$. Assume also that $f(a) = g(a)$ and that $f'(x) > g'(x)$ for all $x \in (a, b)$. Prove that $f(x) > g(x)$ for all $x \in (a, b)$.

(b) Assume that $f, g : (a, b] \to \mathbf{R}$ are continuous, and are differentiable on $(a, b)$. Assume also that $f(b) = g(b)$ and that $f'(x) > g'(x)$ for all $x \in (a, b)$. In this case, what order-relation do $f(x)$ and $g(x)$ obey for $x \in (a, b)$?

In part (b), you are not being asked for a formal proof. Just state how, under the hypotheses in (b), any inequalities you used for the proof in part (a) become modified, and how this affects or does not affect the conclusion.

B5. Prove that

$$\text{(a)} \qquad \frac{x}{1+x^2} < \tan^{-1} x < x \quad \text{for all } x > 0,$$

and

$$\text{(b)} \qquad x < \sin^{-1} x < \frac{x}{\sqrt{1-x^2}} \quad \text{for } \ 0 < x < 1.$$

(Here "$\tan^{-1}$" and "$\sin^{-1}$" are the inverse tangent and inverse sine functions, also known as "arctan" and "arcsin" respectively.)

B6. Prove that, for all $x > 0$,

$$\text{(a)} \quad \sin x < x,$$

$$\text{(b)} \quad \cos x > 1 - \frac{x^2}{2}, \qquad \text{and}$$

$$\text{(c)} \quad x - \frac{x^3}{3!} < \sin x < x - \frac{x^3}{3!} + \frac{x^5}{5!}\ .$$

(Warning: if you try to use Taylor's Theorem for this problem—which is not recommended—don't forget that numbers of the form "$\sin c$" or "$\cos c$" can be negative as well as positive!)

B7. In class we proved that if $I \subset \mathbf{R}$ is an interval, $f : I \to \mathbf{R}$ is continuous on $I$ and differentiable on $I^\circ$, and $f'(x) > 0$ for all $x \in I^\circ$, then $f$ is strictly increasing (i.e. $x_1 < x_2$ implies $f(x_1) < f(x_2)$). In this problem we show that the requirement "$f'(x) > 0$ for <u>all</u> $x \in I^\circ$" can be somewhat relaxed without affecting the conclusion. Parts (a) and (b) draw successively stronger conclusions by using successively weaker hypotheses. Each problem-part is intended to help you do the next part, with the exception that part (d) is independent of all the other parts.

(a) Let $a, b \in \mathbf{R}$, with $a < b$. Let $I$ be any of the intervals $[a, b), (a, b]$, or $[a, b]$. Assume that $f : I \to \mathbf{R}$ is continuous, is differentiable on the open interval $(a, b)$, that $f'(x) \geq 0$ for all $x \in (a, b)$, and that $f'(x) = 0$ for at most finitely many $x \in (a, b)$. Prove that $f$ is strictly increasing on $I$.

(b) Let $I \subset \mathbf{R}$ be a nonempty interval (not necessarily bounded). Assume that $f : I \to \mathbf{R}$ is differentiable and that $f'(x) \geq 0$ for all $x \in I$. Let $Z(f') = \{x \in I \mid f'(x) = 0\}$ (the *zero-set* of $f'$), and assume that $Z(f')$ has no cluster points in in $I$. (Note that if $I$ is not closed, we are not ruling out cluster points in $\overline{I} \setminus I$.) Prove that $f$ is strictly increasing on $I$.

(c) Define $f : \mathbf{R} \to \mathbf{R}$ by $f(x) = x - \sin x$. Prove that $f$ is strictly increasing.

(d) Let $a, b \in \mathbf{R}$, with $a < b$. Assume that $f : [a, b) \to \mathbf{R}$ is continuous, and is strictly increasing on the open interval $(a, b)$. Prove that $f$ is strictly increasing on $[a, b)$. (Note that no differentiability is assumed; this problem-part is independent of the previous parts.) Similarly, prove that if $f : (a, b] \to \mathbf{R}$ is continuous, and is strictly increasing on interval $(a, b)$, then $f$ is strictly increasing on $(a, b]$. As a corollary, deduce that if $f : [a, b] \to \mathbf{R}$ is continuous, and is strictly increasing on $(a, b)$, then $f$ is strictly increasing on $[a, b]$.

B8. Let $I \subset \mathbf{R}$ be an open interval, let $x_0 \in I$, and $f : I \to \mathbf{R}$ be a function that is continuous on $I$ and differentiable on $I \setminus \{x_0\}$. Assume that $\lim_{x \to x_0^+} f'(x)$ and $\lim_{x \to x_0^-} f'(x)$ exist and are equal. Prove that $f$ is differentiable at $x_0$ and that $f'(x_0)$ has the same value as these two limits (and hence that $f'$ not only exists at $x_0$ but is continuous there).

Be careful not to assume that $f$ has any properties not given in the hypotheses. For example, don't assume that $f'$ is continuous on $I \setminus \{x_0\}$.