MAA 4212, Spring 2020—Assignment 4's non-book, non-notes problem

B1. Let $a, b \in \mathbf{R}$, with $a < b$.

(a) For $f, g \in C([a,b])$ (the space of real-valued continuous functions on $[a,b]$) define

$$\langle f, g \rangle = \int_a^b fg. \tag{1}$$

(In Rosenlicht exercise VI.10 you showed that the product of integrable functions is integrable. But even without that exercise, the product of continuous functions is continuous, hence integrable. Thus, by either of these arguments, equation (1) *does* define a real number $\langle f, g \rangle$.)

Let $\langle \cdot, \cdot \rangle$ denote the map $C([a,b]) \times C([a,b]) \to \mathbf{R}$ defined by $(f,g) \mapsto \langle f, g \rangle$. Show that $\langle \cdot, \cdot \rangle$ is an inner product on the (infinite-dimensional) vector space $C([a,b])$).

(b) Since the product of integrable functions is integrable, equation (1) can be used to define a map $\langle \cdot, \cdot \rangle : \mathcal{R}([a,b]) \times \mathcal{R}([a,b]) \to \mathbf{R}$. However, this map is *not* an inner product on the vector space $\mathcal{R}([a,b])$. Why not?