MAA 4212, Spring 2020—Assignment 5's non-book problems

B1. Let $(a_n)_{n=1}^\infty, (b_n)_{n=1}^\infty$ be sequences in $\mathbf{R}$ such that $\sum_n a_n$ converges but $\sum_n b_n$ diverges. Prove that $\sum_{n=1}^\infty (a_n + b_n)$ diverges.

B2. In class we proved the "alternating-series theorem": if the terms of the real-valued sequence (a_n) strictly alternate in sign, and $|a_n|$ decreases *monotonically* to zero, then $\sum a_n$ converges. Give a example showing that the monotonicity assumption in this theorem cannot be removed. (I.e. find a counterexample to the following statement: if the sequence (a_n) strictly alternates in sign, and $\lim_{n\to\infty} a_n = 0$, then $\sum a_n$ converges.) Show that the series in your counterexample does, in fact, diverge.

B3. Let $(a_n)_{n=1}^\infty$ be a sequence in $\mathbf{R}$ for which $\sum_{n=1}^\infty a_n$ is conditionally convergent (i.e. convergent but not absolutely convergent). Prove the following:

(a) There are infinitely many n for which a_n is positive, and infinitely many n for which a_n is negative.

(b) Let $(b_i)_{i=1}^\infty$ be the subsequence of (a_n) consisting of the positive terms of a_n. (I.e. $b_i = a_{n_i}$, where n_i is the index of the i^{th} positive term of (a_n).) Similarly let $(c_i)_{i=1}^\infty$ be the subsequence of (a_n) consisting of the negative terms of a_n. Show that both of the series $\sum_{i=1}^\infty b_i$, $\sum_{i=1}^\infty c_i$ diverge.

(c) Let r be any element of the extended reals $\mathbf{R}_{\text{ext}}$. Prove that there exists a rearrangement of $\sum_{n=1}^\infty a_n$ that converges in $\mathbf{R}_{\text{ext}}$ to r. In other words, prove that (i) for any *real* number r, there exists a bijection $f : \mathbf{N} \to \mathbf{N}$ such that $\sum_{n=1}^\infty a_{f(n)} = r$; (ii) there exists a bijection $f : \mathbf{N} \to \mathbf{N}$ such that $\sum_{n=1}^\infty a_{f(n)}$ diverges to ∞ (which, definition, means that the series *con*verges in $\mathbf{R}_{\text{ext}}$ to ∞, and (iii) there exists a bijection $f : \mathbf{N} \to \mathbf{N}$ such that $\sum_{n=1}^\infty a_{f(n)}$ diverges to $-\infty$.

B4. Let $(a_{(m,n)})_{(m,n)\in\mathbf{N}\times\mathbf{N}}$ be a "doubly indexed sequence" in $\mathbf{R}$—a map $A : \mathbf{N}\times\mathbf{N} \to \mathbf{R}$, where $a_{(m,n)} = A(m,n)$. It is sometimes useful to picture $(a_{(m,n)})$ as an "infinity-by-infinity matrix". In this problem we are interested in attaching meaning to the notation "$\sum_{m,n} a_{(m,n)}$," also written "$\sum_{m,n=1}^\infty a_{(m,n)}$". (Our notation "$a_{(m,n)}$" can also be replaced by any other notation for the values of a function $\mathbf{N}\times\mathbf{N} \to \mathbf{R}$, e.g. $a_{m,n}$ or $A(m,n)$.)

Definition. The doubly-indexed series $\sum_{m,n} a_{(m,n)}$ is *absolutely convergent* (or *converges absolutely*) if there exists a bijection $f : \mathbf{N} \to \mathbf{N}\times\mathbf{N}$ such that $\sum_{j=1}^\infty a_{f(j)}$ is absolutely convergent. (Said more loosely, we are calling the doubly-indexed series absolutely convergent if there is some order in which we can add up the entries of the "infinite matrix" $(a_{(m,n)})$ as the terms of an absolutely convergent singly-indexed series.)

(a) Prove that if $\sum_{m,n} a_{(m,n)}$ converges absolutely and $f, g : \mathbf{N} \to \mathbf{N}\times\mathbf{N}$ are bijections, then $\sum_{j=1}^\infty a_{f(j)} = \sum_{j=1}^\infty a_{g(j)}$. Hence if $\sum_{m,n} a_{(m,n)}$ converges absolutely, we can unambiguously define

$$\sum_{m,n} a_{(m,n)} = \sum_{j=1}^{\infty} a_{f(j)}$$

where f is *any* bijection $\mathbf{N} \to \mathbf{N} \times \mathbf{N}$.

(b) Explain why we should not attach any numerical value (in $\mathbf{R}$) to the notation "$\sum_{m,n} a_{(m,n)}$" if this doubly-indexed series is *not* absolutely convergent. (*Hint*: Problem B3(c).)

(c) Prove that if $\sum_{m,n} a_{(m,n)}$ is absolutely convergent then $\sum_{m=1}^{\infty} a_{(m,n)}$ converges for all $n \in \mathbf{N}$, $\sum_{n=1}^{\infty} a_{(m,n)}$ converges for all $m \in \mathbf{N}$, and

$$\sum_{m,n} a_{(m,n)} = \sum_{m=1}^{\infty} \left(\sum_{n=1}^{\infty} a_{(m,n)} \right) = \sum_{n=1}^{\infty} \left(\sum_{m=1}^{\infty} a_{(m,n)} \right).$$

(d) Let $\sum_{n=1}^{\infty} b_n, \sum_{n=1}^{\infty} c_n$ be absolutely convergent. Prove that $\sum_{m,n} b_m c_n$ is absolutely convergent, and that

$$\sum_{m,n} b_m c_n = \left(\sum_{n=1}^{\infty} b_n \right) \left(\sum_{n=1}^{\infty} c_n \right).$$

Remark. In the absolutely convergent case, enumerating $\mathbf{N} \times \mathbf{N}$ in the order

$$\begin{array}{lll} (1,1) & & \\ (1,2) & (2,1) & \\ (1,3) & (2,2) & (3,1) \\ & & \cdots \end{array}$$

leads us to

$$\sum_{m,n} a_{(m,n)} = \sum_{k=1}^{\infty} \left(\sum_{n+m=k} a_{(m,n)} \right). \qquad (1)$$

One of the main reasons that the conclusions of problem B4 are important is the following application to power series, in which the enumeration scheme in (1) appears naturally. (For power series, we index the terms using $\mathbf{N} \cup \{0\}$ rather than $\mathbf{N}$, but aside from the slight bookkeeping change this clearly makes no difference in the conclusions of B4.) Suppose you are multiplying two polynomials together, say $a_0 + a_1 x + \cdots + a_N x^N$ (i.e. $\sum_{n=0}^{N} a_n x^n$) and $b_0 + b_1 x + \cdots + b_M x^M$ (i.e. $\sum_{m=0}^{M} b_m x^m$). After multiplying out, you generally rewrite the result by grouping together all the terms with a given power of x, which is the finite-series statement

$$\left(\sum_{n=0}^{N} a_n x^n \right) \left(\sum_{m=0}^{M} b_m x^m \right) = \sum_{k=0}^{N+M} \left(\sum_{n+m=k} a_n b_m \right) x^k.$$

Since power series are absolutely convergent on the interiors of their intervals of convergence, parts (a) and (d) imply that on the interior of the smaller of the intervals of convergence of two power series centered at 0, you can multiply the series together just as if they were polynomials (with infinitely many terms). For fun, you might try to show the identity $\sin^2 x + \cos^2 x = 1$ or $\sin x \cos x = \frac{1}{2}\sin(2x)$ or $(e^x)^2 = e^{2x}$ this way.

B5. Here is a True/False test. Note that statements (a) and (b) have a hypothesis that is missing in statements (c) and (d).

(a) If (a_n) is a sequence of *non-negative* real numbers, and $\sum_n a_n$ converges, then $\sum_n a_n^2$ converges.

(b) If (a_n) is a sequence of *non-negative* real numbers, and $\sum_n a_n$ converges, then $\sum_n a_n^3$ converges.

(c) If (a_n) is a sequence of real numbers and $\sum_n a_n$ converges, then $\sum_n a_n^2$ converges.

(d) If (a_n) is a sequence of real numbers and $\sum_n a_n$ converges, then $\sum_n a_n^3$ converges.

Take this True/False test and prove your answers. You have already done a problem that addresses statement (a); this statement is included primarily for purposes of comparison with parts (b) and (c). You will probably find (c) a little more difficult than (a) and (b). You will probably find (d) several orders of magnitude more difficult than (a), (b), or (c). Think of (d) as extra credit rather than as a problem you are expected to be able to solve.

In the two problems below, which I may end up deferring to the next assignment if you find the rest of the current assignment too time-consuming, you are allowed to use your knowledge of trigonometric functions and their derivatives. In problem B7 you will derive a fun fact with which you can impress people at parties.[1]

B6. (a) Let $a, b \in \mathbf{R}, a < b$. Suppose $g : (a, b) \to \mathbf{R}$ is differentiable. Prove that if g' is bounded, then there exists a continuous extension of g to the closed interval $[a, b]$ (i.e. there exists a continuous function $\tilde{g} : [a, b] \to \mathbf{R}$ that coincides with g on (a, b)).

(b) Prove that the integration-by-parts formula (Rosenlicht p. 133/#17) generalizes to the case in which u is replaced by a function g satisfying only the hypotheses in part (a). (In the Rosenlicht problem, u' is assumed to exist and be continuous on an open interval containing $[a, b]$.)

(c) Suppose $g : (0, \pi) \to \mathbf{R}$ is continuously differentiable and that g' is bounded. Prove that

$$\lim_{n\to\infty} \int_0^\pi g(x)\sin(nx)\, dx = 0.$$

(Hint: (a) and (b) come before (c). Suggestion: read part (d) before starting part (c).)

[1] Caution is advised if there's anybody at the party whom you wish to date.

(d) Did part (c) involve interchanging the order of a limit as $n \to \infty$ and an integral? Would such an interchange even have been possible? Why or why not?

B7. In this problem, you are free to use the conclusion of the previous problem.

As you saw in Rosenlicht problem VII.10, the integral test (Rosenlicht problem VII.9) implies that $\sum_{n=1}^{\infty} 1/n^p$ (called the *p-series*) converges for all real $p > 1$. The proof of the integral test can be used to give crude upper and lower bounds on the sum, but not its exact value. In this problem you will end up computing the exact value of $\sum 1/n^2$, by roundabout means.

(a) Let $f : [0, \pi] \to \mathbf{R}$ be a continuous function. Suppose that $f(0) = f(\pi) = 0$, and that f'' exists on $(0, \pi)$ and extends to a continuous function on $[0, \pi]$. For $0 < x < \pi$, define $g(x) = f(x)/\sin(x)$. Prove that the limit of g' exists at both endpoints of $[0, \pi]$, and hence that g' extends to a continuous (and therefore bounded) function on $[0, \pi]$.

Note: this problem is one of those rare instances in which even a real mathematician might use l'Hôpital's Rule.

(b) Let f be as in part (a). Prove that

$$\lim_{n\to\infty} \int_0^{\pi} f(x) \frac{\sin(nx)}{\sin(x)}\, dx = 0.$$

(c) Verify that if n is any integer, then

$$\int_0^{\pi} x(\pi - x)\cos(2nx)\, dx = \begin{cases} -\pi/(2n^2), & n \neq 0 \\ \pi^3/6, & n = 0 \end{cases}.$$

(Note: for $n \neq 0$ the computation is simpler if you **do not** break the integral up into two pieces, one for $x^2 \cos 2nx$ and $x \cos 2nx$.) Use this to prove that

$$\sum_{n=1}^{\infty} \left(\int_0^{\pi} x(\pi - x)\cos(2nx)\, dx \right) = -\frac{\pi}{2} \sum_{n=1}^{\infty} \frac{1}{n^2}.$$

(d) Show that for all integers $n \geq 1$,

$$\cos(2x) + \cos(4x) + \cos(6x) + \cdots + \cos(2nx) = \frac{1}{2} \left(\frac{\sin((2n+1)x)}{\sin(x)} - 1 \right).$$

Use this to prove that

$$\sum_{n=1}^{\infty} \left(\int_0^{\pi} x(\pi - x)\cos(2nx)\, dx \right) = -\frac{1}{2} \int_0^{\pi} x(\pi - x) dx.$$

(e) Using the work above, determine the exact value of $\sum_{n=1}^{\infty} \frac{1}{n^2}$.