MAA 4212, Spring 2021—Assignment 1's non-book problems

B1. Let (X, d) be a metric space.

(a) Prove the "iterated triangle inequality": for all $n \geq 1$ and all x, z, and $y_1, y_2, \ldots, y_n \in X$, we have

$$d(x,z) \leq d(x,y_1) + d(y_1,y_2) + \cdots + d(y_{n-1},y_n) + d(y_n,z).$$

(b) Prove that for all $x, y, z \in X$, we have

$$|d(x,y) - d(x,z)| \leq d(y,z).$$

B2. Let X be any nonempty set and let $\vec{y} := (y_n)_{n=1}^{\infty}$ be a sequence in X. Recall that we say that $\vec{y}$ is *eventually constant* if there exists $p \in X$ and $N \in \mathbf{N}$ such that for all $n \geq N$, we have $y_n = p$.

Note: Part (a), a reworded version of Tao exercise 1.1.13, is a special case of part (b), and is not needed in order to do part (b). If you are able to do (b) without (a) first, you don't need to do (a) separately.

(a) Show that $\vec{y}$ converges in (X, d_{disc}) if and only if $\vec{y}$ is eventually constant. Also show that in the convergent case, the limit is the eventual value of the sequence (the "p" above).

(b) Same as part (a), but with d_{disc} replaced by an arbitrary *discrete* metric (as defined in class).

B3. Let V be a vector space with more than one element, and let d be a metric on V such that d is a bounded function. (The metric d_{disc} is one such example.) Show that there is no norm $\| \ \|$ on V such that $d = d_{\| \ \|}$.

B4. Let X be a set, and let d_1, d_2 be metrics on X. Recall that (in this course) we say that d_1 is *equivalent* to d_2, and write "$d_1 \sim d_2$", if there exist $c_1, c_2 > 0$ such that for all $x, y \in X$, we have

$$d_2(x,y) \leq c_1 d_1(x,y) \qquad \text{and} \qquad d_1(x,y) \leq c_2 d_2(x,y). \tag{1}$$

(a) Show that the following are equivalent:

(i) $d_1 \sim d_2$.

(ii) There exist $c_1, c_2 \in \mathbf{R}$ such that for all $x, y \in X$, inequalities (1) hold.

(iii) There exist $c_3 \in \mathbf{R}$ and $c_4 > 0$ such that for all $x, y \in X$, we have $c_4 d_1(x,y) \le d_2(x,y) \le c_3 d_1(x,y)$.

(iv) There exists $C > 0$ such that for all $x, y \in X$, we have $\frac{1}{C} d_1(x,y) \le d_2(x,y) \le C d_1(x,y)$.

(b) Show that the relation "$\sim$" above is an equivalence relation on the set of all metrics on X.

(As a consequence of part (b), since the relation is symmetric, if $d_1 \sim d_2$ we may sensibly say "d_1 and d_2 are equivalent.")

B5. Let $n \ge 1$. Show that the ℓ^1, ℓ^∞, and ℓ^2 norms on $\mathbf{R}^n$ are equivalent, and hence that their associated metrics are equivalent. (This is an amalgamation of Tao exercises 1.1.8 and 1.1.10.)

B6. Let d_1, d_2 be equivalent metrics on a set X. For $i \in \{1, 2\}$, let us say that sequence in X is *d_i-convergent* if the sequence converges in the metric space (X, d_i).

Let $(x_n)_{n=1}^\infty$ be a sequence in X.

(a) Let $p \in X$. Prove that $x_n \underset{d_1}{\longrightarrow} p$ if and only if $x_n \underset{d_2}{\longrightarrow} p$.

(b) Deduce from part (a) that a sequence in X is d_1-convergent if and only if it is d_2-convergent.

Note: The terminology "d_1-convergent", "d_2-convergent" is not standard. However, if you use this terminology (when given two metrics on the same set) outside this class, any mathematician will assume you are using it with the meaning above.

B7. Let $\mathbf{R}^\infty$ denote the space of real-valued sequences (indexed by $\mathbf{N} = \{1, 2, 3, \dots\}$). Notation for this problem: a sequence $(x_n)_{n=1}^\infty$, is also denoted $\vec{x}$, and conversely, if the notation $\vec{x}$ is used for a sequence, then the n^{th} term of the sequence is named x_n. (All of this applies with x replaced by other letters as well.)

Just as last semester (modulo notation), for $\vec{x}, \vec{y} \in \mathbf{R}^\infty$ and $c \in \mathbf{R}$, we define $\vec{x} + \vec{y} = (x_n + y_n)_{n=1}^\infty$ and $c\vec{x} = (cx_n)_{n=1}^\infty$, and define $\vec{0} \in \mathbf{R}^\infty$ to be the constant sequence all of whose terms are 0. In homework last semester (in my section), you saw that with these operations and zero-element, $\mathbf{R}^\infty$ is a vector space.

Define $\ell^\infty(\mathbf{R}) \subseteq \mathbf{R}^\infty$ to be the set of all *bounded* real-valued sequences, and define $\ell^1(\mathbf{R}) \subseteq \mathbf{R}^\infty$ by

$$\ell^1(\mathbf{R}) := \{\vec{x} \in \mathbf{R}^\infty : \sum_{n=1}^\infty |x_n| \text{ converges}\}.$$

(a) Show that $\ell^1(\mathbf{R}) \subsetneq \ell^\infty(\mathbf{R})$.

(b) Show that $\ell^1(\mathbf{R})$ and $\ell^\infty(\mathbf{R})$ are vector subspaces of $\mathbf{R}^\infty$. (It then follows automatically from (a) that $\ell^1(\mathbf{R})$ is a vector subspace of $\ell^\infty(\mathbf{R})$.)

(*Recall from class*: The term "vector subspace" means exactly what you simply called "subspace" in MAS 4105. In this class, we need to be careful not to use the word *subspace* ambiguously, since we have also defined "*(metric) subspace*" of a metric space (X, d), meaning the metric space $(Y, d|_{Y\times Y})$, where Y is any subset of X.)

(c) Define functions $\| \; \|_\infty : \ell^\infty(\mathbf{R}) \to \mathbf{R}$ and $\| \; \|_1 : \ell^1(\mathbf{R}) \to \mathbf{R}$ as follows:

$$\begin{aligned} \|\vec{x}\|_\infty &= \sup\{|x_n| : n \in \mathbf{N}\} \quad \text{for all } \vec{x} \in \ell^\infty(\mathbf{R}), \\ \|\vec{x}\|_1 &= \sum_{n=1}^{\infty} |x_n| \quad \text{for all } \vec{x} \in \ell^1(\mathbf{R}). \end{aligned}$$

Show that $\| \; \|_\infty$ and $\| \; \|_1$ are norms on the vector spaces $\ell^\infty(\mathbf{R})$ and $\ell^1(\mathbf{R})$, respectively.

Notation convention. The norms defined above are called the ℓ^∞ and ℓ^1 norms on $\ell^\infty(\mathbf{R})$ and $\ell^1(\mathbf{R})$, respectively; their associated metrics d_{ℓ^∞} and d_{ℓ^1} are called the ℓ^∞ and ℓ^1 metrics on these vector spaces. These norms and metrics are regarded as the "standard" norms and metrics on these vector spaces. When $\ell^\infty(\mathbf{R})$ or $\ell^1(\mathbf{R})$ is referred to as a *metric* space, or treated implicitly as one, and no norm or metric is stated explicitly, the corresponding "standard" norm and metric are being assumed implicitly. Thus, the notations $\ell^\infty(\mathbf{R})$ or $\ell^1(\mathbf{R})$ are sometimes used just for the underlying sets defined earlier in this problem, and sometimes used for particular metric spaces in which these are the underlying sets. Context tells you which meaning is intended (if the writer has done his or her job). For example, in the statement "$\ell^1(\mathbf{R}) \subsetneq \ell^\infty(\mathbf{R})$", the symbol "$\subsetneq$" makes clear that we are regarding $\ell^\infty(\mathbf{R})$ and $\ell^1(\mathbf{R})$ simply as *sets* in this statement.

(d) Since $\ell^1(\mathbf{R}) \subseteq \ell^\infty(\mathbf{R})$, we can consider the metric subspace $(\ell^1(\mathbf{R}), d_{\ell^\infty})$. (To avoid losing readers in a notational forest, I've just written "d_{ℓ^∞}" here, rather than the more precise "$d_{\ell^\infty}|_{\ell^1(\mathbf{R})\times\ell^1(\mathbf{R})}$").

(i) Show that for all $\vec{x} \in \ell^1(\mathbf{R})$ we have $\|\vec{x}\|_{\ell^\infty} \le \|\vec{x}\|_1$, and hence that any sequence in $\ell^1(\mathbf{R})$ that is d_{ℓ^1}-convergent is also d_{ℓ^∞}-convergent. (Note: "*sequence* in $\ell^1(\mathbf{R})$" does not mean "*element* of $\ell^1(\mathbf{R})$." A sequence in $\ell^1(\mathbf{R})$, or more generally in $\mathbf{R}^\infty$, is a *sequence of sequences*.)

(ii) Show that there exists *no* $c \in \mathbf{R}$ such that for all $\vec{x} \in \ell^1(\mathbf{R})$, we have $\|\vec{x}\|_{\ell^1} \le c\|\vec{x}\|_\infty$.

What (i) and (ii) show, together, is that only one of the two inqualities in the definition of "equivalent norms" holds for the above norms on $\ell^1(\mathbf{R})$ (in contrast to what problem B5 shows for the norms on $\mathbf{R}^n$ carrying the same names). The ℓ^∞ and ℓ^1 norms on $\ell^1(\mathbf{R})$ are *not* equivalent, and therefore neither are their associated metrics.

(e) Find (with proof) a sequence $\vec{x}$ in $\ell^1(\mathbf{R}) \subseteq \ell^\infty(\mathbf{R})$ that is d_{ℓ^∞}-convergent but not d_{ℓ^1}-convergent. (Original problem had a typo, "sequence $\vec{x} \in \ell^1(\mathbf{R})$" instead of "sequence in $\ell^1(\mathbf{R})$". This made the problem-statement make no sense, since an *element* of a metric space (X, d) isn't something to which the term "d-convergent" can apply. Because of that, I treated this problem-part as extra credit.)

Hint. In B5, for each $n \in \mathbf{N}$ you (should have) found a constant $c(n)$ such that for all $v \in \mathbf{R}^n$, we have $\|v\|_{\ell^1} \leq c(n)\|v\|_{\ell^\infty}$, where the indicated norms are the ℓ^1 and ℓ^∞-norms <u>on $\mathbf{R}^n$</u>. If the constants $c(n)$ you found are valid, the sequence $(c(n))_{n=1}^\infty$ grows without bound. In finding $c(n)$, you probably found "worst case" vectors $v \in \mathbf{R}^n$ for which no constant smaller than your $c(n)$ would have worked. These "worst case" vectors should give you an idea how to construct a sequence $(\vec{y}^{(m)})_{m=1}^\infty$ in $\ell^1(\mathbf{R})$ that converges in $\ell^\infty(\mathbf{R})$, but for which the real-valued sequence $(\|\vec{y}^{(m)}\|)_{m=1}^\infty$ is unbounded. You should be able to use this unboundedness to show that $(\vec{y}^{(m)})_{m=1}^\infty$ cannot converge in $\ell^1(\mathbf{R})$.

B8. Considering $\mathbf{Q}$ (the set of rational numbers) as a subset of the metric space $\mathbf{R}$, find the boundary $\partial\mathbf{Q}$ and closure $\overline{\mathbf{Q}}$.