MAA 4212, Spring 2021—Assignment 2's non-book problems

B1. (Proving Proposition 5.6(c)(ii) of the lecture notes.) Let (X, d) be a metric space, $x_0 \in X$, and $r \geq 0$. Prove that the "closedball" $\overline{B}(x_0, r)$ is a closed subset of X.

(Once this is proved, the conventional two-word terminology "closed ball" is safe to use without fear of ambiguity.)

B2. Show that, in $\mathbf{R}$, every open (respectively closed) interval, as defined last semester, is an open (resp. closed) ball, and hence an open (resp. closed) subset of $\mathbf{R}$.

B3. Let $S = \{\frac{1}{n} : n \in \mathbf{N}\} \subseteq \mathbf{R}$.

(a) Show that S is neither open nor closed in $\mathbf{R}$.

(b) Show that $S \cup \{0\}$ is closed in $\mathbf{R}$.

B4. (a) Let X be any set. Show that *every* subset of X is both open and closed in the discrete metric space (X, d_{disc}).

Note: Part (a) is a special case of part (b), and is not needed in order to do part (b). If you are able to do (b) without (a) first, you don't need to do (a) separately.

(b) Same as part (a), but with d_{disc} replaced by an arbitrary *discrete* metric (as defined in class).

B5. Considering the metric space $(\mathbf{Q}, d)$, where $d(x, y) = |x - y|$. Give an example, with proof, of a nonempty, proper subset of $(\mathbf{Q}, d)$ that is both open and closed in this metric space. (Said another way: considering $\mathbf{Q}$ as a subset of $\mathbf{R}$, where $\mathbf{R}$ is given the standard metric, you're looking for a nonempty, proper subset of $\mathbf{Q}$ that is both relatively open with respect to $\mathbf{Q}$ and relatively closed with respect to $\mathbf{Q}$.)

Note: Do not expect your subset to be either open or closed in $\mathbf{R}$, let alone *both* open and closed in $\mathbf{R}$. As we will see in the not-too-distant future, there is no nonempty, proper subset of $\mathbf{R}$ that is both open and closed with respect to the standard metric.

B6. Let $(V, \| \ \|)$ be a normed vector space, viewed as a metric space with the associated metric. Show that for all $v \in V$,

$$\text{for each } r > 0 \text{ we have} \quad B(v, r) \quad = \quad \{v + w \mid w \in B(0, r)\},$$

and

$$\text{for each } r \geq 0 \text{ we have} \quad \overline{B}(v, r) \quad = \quad \{v + w \mid w \in \overline{B}(0, r)\}.$$

In other words, each open (respectively, closed) ball centered at a given v is simply the translation, by v, of the open (respectively, closed) ball of the same radius centered at the origin. *Note*: In each of the displayed statements above, the symbol "0" has two different meanings: the real number "zero" and the zero element of V (which you have sometimes seen me denote 0_V). You are expected to be able to tell, from context, what each "0" in the displayed statements is.[1]

B7. (Proving Proposition 5.6(h) in the lecture notes; equivalently, Proposition 1.2.15(h) in Tao.)

Let (X, d) be a metric space, and let $E \subseteq X$.

(a) Prove that $\text{int}(E)$ is the largest open subset of E; i.e. that

- $\text{int}(E)$ is open, and
- if $V \subseteq E$ and V is open (in X), then $V \subseteq \text{int}(E)$.

(b) Prove that $\overline{E}$ is the smallest closed subset of E; i.e. that

- $\overline{E}$ is closed, and
- if $K \supseteq E$ and K is closed (in X), then $\overline{E} \subseteq K$.

B8. (a) Let $Y = [1, 3] \subseteq \mathbf{R}$. Show that the set $E = [1, 2)$ is relatively open with respect to Y.

(b) Let $Z = (4, 6) \subseteq \mathbf{R}$. Show that the set $F = [5, 6)$ is relatively closed with respect to Z.

B9. Let d_1 and d_2 be two metrics on a nonempty set X.[3] Call a set $U \subseteq X$ "d_1-open" if U is open in the metric space (X, d_1), and "d_2-open" if U is open in the metric space (X, d_2). For $i \in \{1, 2\}$, analogously define "d_i-Cauchy sequence" and "d_i-bounded subset," and define X to be "d_i-complete" (respectively, "d_i-compact") if the metric space (X, d_i) is complete (resp., compact).

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[1]Usually, allowing the same notation to have two different meanings in the same sentence (or paragraph, proof, etc.) is a terrible idea, deserving of a bad-writing penalty. The multiple-meanings use of "0" is an exception to this rule, and you'll find "0" used this way by most mathematicians and in most textbooks. One reason is that there's a zero element of every field, every vector space, and, more generally, every abelian group[2]; using different notation for each zero-element can lead to hard-to-read clutter. Another reason for making this exception is that "0" isn't usually a symbol you *introduce*; you treat it as having already been introduced, for every algebraic structure that has an element called "zero", prior to your having started writing. Nonetheless, *sometimes*, using different notation for different zero-elements is nearly essential to prevent confusion.

[3]Everything you'll prove in this problem is true even if X is empty; I'm just not asking you to spend time to deal with this trivial case.

(a) Suppose that there exists $c > 0$ such that $d_2(p,q) \leq c d_1(p,q)$ for all $p, q \in X$. Prove that every d_2-open subset of X is d_1-open. *Suggestion*: First, for any $p \in X$ and $r_1 > 0$, show (by an explicit formula) that there exists $r_2 > 0$ such that $B_{(X,d_1)}(p,r_1) \subseteq B_{(X,d_2)}(p,r_2)$. Then use the characterization of open sets in terms of open balls (part (a) of Proposition 5.2 in the lecture notes, or Proposition 1.2.15 in Tao), rather than Tao's definition of open sets (the one given in class).

(b) Prove statements (i)–(vii) below about equivalent metrics. For each of these statements, it is possible to word your argument in such a way that one direction of the "iff" statement actually implies both directions, making it unnecessary to write out nearly-identical proofs for the two directions.

There are several different orders in which the listed facts can be proven; some facts on the list can be deduced from others (possibly with the aid of results proven in class).

(i) Equivalent metrics determine the same open sets and the same closed sets. I.e. if d_1 and d_2 are equivalent and $U \subseteq X$, then U is d_1-open iff U is d_2-open, and U is d_1-closed iff U is d_2-closed.

(ii) Equivalent metrics determine the same Cauchy sequences. I.e. if metrics d_1 and d_2 on X are equivalent and $\vec{x} := (x_n)_{n=1}^{\infty}$ is a sequence in X, then $\vec{x}$ is d_1-Cauchy iff $\vec{x}$ is d_2-Cauchy.

(iii) Equivalent metrics determine the same bounded sets. I.e. if metrics d_1 and d_2 on X are equivalent and $Y \subseteq X$ is d_1-bounded, then Y is d_2-bounded.

(iv) Equivalent metrics determine the same *totally* bounded sets. I.e. if metrics d_1 and d_2 on X are equivalent and $Y \subseteq X$ is totally bounded with respect to d_1 (i.e. as a subset of (X, d_1)), then Y is totally bounded with respect to d_2

(v) If d_1 and d_2 are equivalent metrics on X, then (X, d_1) is sequentially compact iff (X, d_2) is sequentially compact.

(vi) If d_1 and d_2 are equivalent metrics on X, then X is d_1-complete iff X is d_2-complete.

(vii) If d_1 and d_2 are equivalent metrics on X, then X is d_1-compact iff X is d_2-compact.

B10. Let (E, d) be a metric space, $p \in E$, and $r > 0$. Let $S(p,r) = \{q \in E : d(q,p) = r\}$ (the *sphere* of radius r centered at p).

(a) Prove that $\partial(B(p,r)) \subseteq S(p,r)$ (i.e. that the boundary of the ball is contained in the sphere).

(b) Prove that $\overline{B}(p,r)) = \overline{B(p,r)}$ if and only if $\partial(B(p,r)) = S(p,r)$.

(c) Give, with proof, an example of a nonempty, non-singleton metric space in which every sphere of sufficiently small radius is empty.

(d) Give, with proof, an example of a metric space in there is some point p and radius $r > 0$ for which $S(p, r) \neq \emptyset$ but for which $\partial(B(p, r)) = \emptyset$. (The same metric space you used in part (c) should work!)

This shows that, in a general metric space, the boundary of $B(p, r)$ need not be the whole sphere $S(p, r)$. In view of part (b), this also shows that, in a general metric space, the closure of an open ball $B(p, r)$ need not be the closed ball $\overline{B}(p, r)$.

B11. Let $\mathbf{E}^2 = (\mathbf{R}^2, d_{\mathrm{Euc}}) = (\mathbf{R}^2, d_{\ell^2})$, Euclidean 2-space. Let $p \in \mathbf{E}^2$ and let $r > 0$.

(a) Show that $\partial(B(p, r)) = S(p, r)$ (see problem B10). Of course, in $\mathbf{E}^2$, spheres are what we usually call circles!

Hint: To show that an open ball $B(q, s)$ contains a point of $B(p, r)$, or that it contains a point of the complement $\mathcal{C}(B(p, r)) := \mathbf{E}^2 \setminus B(p, r)$, consider which points of the ray $\{p + t(q - p) : t \in [0, \infty)\}$ lie in $B(p, r) \cap B(q, s)$, or lie in $\mathcal{C}(B(p, r)) \cap B(q, s)$.[4] Drawing a picture will lead you to the right *conjecture*, but you will have to prove algebraically that your conjecture is correct.

(Remember that there is no such thing as "proof by picture". If you are asserting, for example, that a certain open ball contains points of some other set, you have to *prove* that assertion, not merely assert that it's true because it looks that way in a picture you've drawn. The latter exemplifies "proof by lack of imagination": you believe that some fact is true simply because you can't think of how it could be false, and then you assert that that fact *is* true without supplying any argument that relies only on the given hypotheses and previously proven facts.)

(b) Show that $\overline{B(p, r)} = \overline{B}(p, r)$ (i.e. the closure of an open ball is the closed ball with the same center and radius). *Note*: As you saw in problem B10, there are metric spaces in which this is not true!

(c) Show that the closed ball $\overline{B}(p, r)$ is not an open set.

(d) Re-do parts (a)–(c) with $\mathbf{E}^2$ replaced by an arbitrary normed vector space $(V, \| \, \|)$. Once (a)–(c) are done, you should find this easy; if not, then your arguments in (a)–(c) are probably wrong.

B12. Show that "(clsd $\subseteq$ seq. cpt.) implies seq. cpt)": If (X, d) is a sequentially compact metric space, and $Y \subseteq X$ is a closed subset, then $(Y, d|_{Y \times Y})$ is sequentially compact.

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[4]For students using LaTeX: to get the symbol "$\mathcal{C}$", use `{\mathcal C}`.