MAA 4212, Spring 2021—Assignment 3's non-book problems

I originally wrote problems B1 and B2 before looking at Tao's exercises for Section 2.1. Problem B1 is essentially the same as Tao's exercise 2.1.5; problem B2(c) is essentially the same as Tao's exercises 2.1.6–2.1.7.

B1. ("Inclusion maps are continuous.") Let (X, d) be a metric space and let $S \subseteq X$. Define $\iota_S : S \to X$ by: $\iota_S(x) = X$ for all $x \in S$. Prove that ι_S is continuous as a map from $(S, d|_{S\times S})$ to (X, d).

Note: Unless $S = X$, we do not call ι_S an *identity map*, since the domain and codomain are not the same set. However, if $S = X$, then the inclusion map ι_S *is* the identity map of (X, d_X). So a corollary of what you are proving for inclusion maps is that the *identity map on a metric space is continuous.*

FYI: "ι" is the lower-case Greek letter *iota*. The LaTeX command for it is \iota.

Additional note In Tao's notation in exercise 2.1.5, my "ι_S" would be "$\iota_{S\to X}$". Compared to "ι_S", the notation "$\iota_{S\to X}$" has both an advantage and a disadvantage: the latter notation carries more information, but the extra clutter can be distracting. In situations where I think the more-information advantage outweighs the clutter disadvantage, there is a third notation that has an even greater advantage, and that I may sometimes use: "$\iota_{S\hookrightarrow X}$". (Notation of the form "$f : A \hookrightarrow B$" is often used when a map $f : A \to B$ is injective, which inclusion maps always are. The LaTeX command for "$\hookrightarrow$" is \hookrightarrow.)

B2. Let $(X, d_X), (Y, d_Y)$ be metric spaces, and let $f : X \to Y$ be a function.

(a) ("Restrictions of continuous functions are continuous.") Let $S \subseteq X$. Give two proofs that if f is continuous, then $f|_S$ is continuous (as a map from $(S, d_X|_{S\times S})$ to (Y, d_Y)). For one of these proofs, argue straight from the definition of continuity. For the other proof, start by comparing the map $f|_S$ to the composition $f \circ \iota_S$, where the inclusion map ι_S is as in problem B1. For the other proof, don't directly use anything about composition.

(b) Let $T \subseteq Y$ be a subset that contains the range of f. Define $\hat{f} : X \to T$ by: $\hat{f}(x) = f(x)$ for all $x \in S$. Prove that f is continuous (as a map from (X, d_X) to (Y, d_Y)) iff $\hat{f}$ is continuous as a map from (X, d_X) to $(T, d_Y|_{T\times T})$).

(c) Let $x_0 \in X$. Prove that the conclusions of (a) and (b) hold also if "continuous" is replaced by "continuous at x_0." (You may have already done this, depending on your method of proof for (a) and (b).)

B3. Let X and Y be sets. Let d_X and d'_X be equivalent metrics on X, and let d_Y and d'_Y be equivalent metrics on Y. Let $f : X \to Y$ be a function. Show that f is continuous as a map from (X, d_X) to (Y, d_Y) iff f is continuous as a map from (X, d'_X) to (Y, d'_Y).

B4. Let (X, d) be a metric space, and let $(E_\alpha)_{\alpha \in A}$ be an indexed collection of connected subsets of X (i.e. E_α is connected for each $\alpha \in A$). Suppose there exists $\alpha_0 \in A$ such that for each $\alpha \in A$, the intersection $E_\alpha \cap E_{\alpha_0}$ is non-empty. Show that $\bigcup_{\alpha \in A} E_\alpha$ is connected.

(This is stronger than Tao Exercise 2.4.6, since above we assume only that *one* of the sets in the collection $(E_\alpha)_{\alpha \in A}$ intersects all the others; we do not assume that there is point common to *all* the sets in tthe collection. If the index-set A contains at least three elements, say $\alpha_1, \alpha_2, \alpha_3$, it's easy to construct examples in which the each intersection $E_{\alpha_i} \cap E_{\alpha_j}$ is non-empty $(i, j, \in \{1, 2, 3\})$, but $E_{\alpha_1} \cap E_{\alpha_2} \cap E_{\alpha_3} = \emptyset$.)

B5. Let (X, d) be a metric space and let $x_0 \in X$. Show that x_0 is an isolated point of X if and only if there exists $r > 0$ such that $B(x_0, r) = \{x_0\}$.

B6. Let (X, d) be a metric space and let $E \subseteq X$.

(a) Let $x_0 \in E$. Show that x_0 is a cluster point of E if and only if x_0 is a non-isolated point of $\overline{E}$.

(b) One direction of the "iff" in part (a) shows is that every cluster point of E is an adherent point of E. Use the other direction to show that an adherent point x_0 of E is a cluster point of E if and only if either (i) $x_0 \notin E$ or (ii) x_0 is a non-isolated point of E.

(c) Show that $\overline{E} = E \cup \{\text{all cluster points of } E\}$.

B7. Let $(X, d_X), (Y d_Y)$ be metric spaces, $x_0 \in X$ a cluster point of X, and $f : X \backslash \{x_0\} \to Y$ a function.

(a) Show that f has at most one limit at x_0. (Hence the notations $\lim_{x \to x_0} f(x)$ and $\lim_x f$ are well-defined when a limit exists, and we can phrase non-existence of a limit by saying "$\lim_{x \to x_0} f(x)$ does not exist" or "$\lim_{x_0} f$ does not exist.")

(b) Let $L \in Y$. Let $\tilde{f} : X \to Y$ be the extension of f defined by

$$\tilde{f}(x) = \begin{cases} f(x) & \text{if } x \neq x_0, \\ L & \text{if } x = x_0. \end{cases}$$

Show that $\lim_{x \to x_0} f(x)$ if and only if $\tilde{f}$ is continuous at x_0.

(c) Show that if $g : X \to Y$ is a function, then g is continuous at x_0 if and only if $\lim_{x \to x_0} g(x) = g(x_0)$.

B8. Let $f : (X, d_X) \to (Y, d_Y)$ be a map between metric spaces. We say that f is *Lipschitz* if there exists $K > 0$ such that

$$d_Y(f(x_1), f(x_2)) \leq K d_X(x_1, x_2) \quad \text{for all } x_1, x_2 \in X. \tag{1}$$

Show that if f is Lipschitz, then f is uniformly continuous.

(The converse is false. As students from my fall section should recall: we made the same definitiion of "Lipschitz" last semester just for the case $(Y, d_Y) = \mathbf{R}$ and $X =$ interval $\subseteq \mathbf{R}$. In that context, you showed that "Lipschitz implies uniformly continuous," but we saw that the square-root map from $[0, \infty)$ to $\mathbf{R}$ is uniformly continuous but not Lipschitz.) Note: a constant K for which (1) holds is called a *Lipschitz constant* for the map f. Lipschitz constants are never unique; if K is a Lipschitz constant for f, then so is any $K' > K$.

B9. Let (X, d) be a metric space, $S \subseteq X$ a nonempty subset. For $y \in X$ the *distance from y to S*, which we will write as $\text{dist}(y, S)$, is defined to be $\inf\{d(y, x) : x \in S\}$.

(a) Let $y \in X$. Prove that $\text{dist}(y, S) = 0$ if and only if $y \in \overline{S}$.

(b) Assume S is compact and let $y_0 \in X$. Prove that there exists $x_0 \in S$ such that $d(y_0, x_0) = \text{dist}(y_0, S)$.

(In other words, you are proving that there exists a point of S that, among all points of S, is *closest to* y_0. Note that we do not call y_0 *the* closest point in S to y_0 without unless we know that there is a *unique* such point.)

(c) Assume only that S is *closed* (but still nonempty) but that $(X, d) = \mathbf{E}^n = (\mathbf{R}^n, d_{\text{Euc}})$ (where $n \in \mathbf{N}$ is arbitrary). Again let $y_0 \in X$. Prove the same result as in (b): there exists $x_0 \in S$ such that $d(y_0, x_0) = \text{dist}(y_0, S)$.

B10. Recall that for any function $f : A \to B$, the *graph* of f is the subset of $A \times B$ defined by

$$\begin{aligned} \text{graph}(f) &:= \{(x, f(x)) : x \in A\} \\ &= \{(x, y) \in A \times B : y = f(x)\}. \end{aligned}$$

(a) Let (X, d_X) and (Y, d_Y) be metric spaces, and let $f : X \to Y$ be a continuous function. Let $d_{\max}$ be the metric on $X \times Y$ constructed from d_X and d_Y as in class (p. 12.6 of the lecture notes). Show that $\text{graph}(f)$ is a closed subset of $(X \times Y, d_{\max})$.

(b) As a corollary of part (a), show that if $f : \mathbf{R} \to \mathbf{R}$ is continuous, the graph of f is a closed subset of $\mathbf{E}^2 := (\mathbf{R}^2, d_{\text{Euc}})$.

(c) More generally, show that if $X \subseteq \mathbf{R}$ and $f : X \to \mathbf{R}$ is continuous, then the graph of f is relatively closed with respect to $X \times \mathbf{R} \subseteq \mathbf{E}^2$.

(d) Define $f : \mathbf{R} \to \mathbf{R}$ by $f(x) = \begin{cases} 1/x & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$ Obviously f is not continuous. Show that, nonetheless, its graph is a closed subset of $\mathbf{E}^2$. This shows that the converse of "If $f : X \to Y$ is continuous, its graph is closed" is false.

Hint: consider the function $g : \mathbf{R}^2 \to \mathbf{R}$ defined by $g(x, y) = xy$, and look back at problem 4a on the first midterm. If, instead, you try to show that the complement of the graph(f) is open by using an idea like "the distance from a point in the complement to a closest point in the graph,", you'll be engaging in circular reasoning; see problem B9(c).

B11. Let (X, d) be a metric space. For any $A \subseteq X$, let $\chi_{_A} : X \to \mathbf{R}$ be the characteristic function of A (as a subset of X); recall that this means $\chi_{_A}(x) = \begin{cases} 1 & \text{if } x \in A, \\ 0 & \text{if } x \in X \backslash A. \end{cases}$ (FYI: "χ" is the lower-case Greek letter χ.) Prove that the set of points at which $\chi_{_A}$ is not continuous is precisely ∂A.

B12. Let (X, d_X) and (Y, d_Y) be metric spaces, and suppose that A and B are *closed* subsets of X for which $X = A \cup B$. Let $f : X \to Y$ be a function for which the restrictions $f|_A : A \to Y$ and $f|_B : B \to Y$ are continuous. Prove that f is continuous.

> (An instance of this you've seen before, but that can be proven more easily than the general case above: Suppose $X = Y = \mathbf{R}$, suppose $c \in \mathbf{R}$, let $A = (-\infty, c]$ and $B = [c, \infty)$, and let $f : \mathbf{R} \to \mathbf{R}$ be a function for which $f|_A$ and $f|_B$ are continuous. Then A and B are closed and $A \cup B = \mathbf{R}$, so proble Note that, as in the given problem, A and B are closed and $A \cup B = \mathbf{R}$. In this "toy" problem with $X = \mathbf{R}$ and $A = (-\infty, c]$ and $B = [c, \infty)$, all the work goes into showing that f is continuous at c. For this, we use the continuity of $f|_A$ and $f|_B$ to show that $\lim_{x \to c-} f(x) = \lim_{x \to c+} f(x) = f(c)$. By homework problem from last semester, this implies that $\lim_{x \to c} f(x)$ exists and equals $f(c)$., i.e. that f is continuous at c.
>
> Note that had we taken $A = (-\infty, c)$ instead of $A = (-\infty, c]$, and had still taken $B = [c, \infty)$, we would still have had $\mathbf{R} = A \cup B$, but the continuity of $f|_A$ and $f|_B$ would *not* have implied that f is continuous at c. For example, the characteristic function of the interval $[c, \infty)$ restricts to the constant function 0_{fcn} on $(-\infty, c)$ [our new A] and to the constant function 1_{fcn} on $[c, \infty)$ [our new and old B]. Thus $f|_A$ and $f|_B$ are continuous, but f is not continuous at c. This example shows that, in problem B12, the requirement that both A and B be closed is essential; any proof-attempt that doesn't use the closedness of both sets cannot be correct.)

B13. In this problem you will prove an important, fundamental result:

> **Theorem A3.1**: *For any $n \geq 1$, all norms on $\mathbf{R}^n$ are equivalent. (Thus, their associated metrics are also equivalent.)*

The proof is a beautiful application of the Heine-Borel Theorem and the Extreme Value Theorem. Below is a sketch of the argument. Use this sketch to write out a complete proof. (Your proof should not explicitly refer to anything like "Step m given in the sketch;" the sketch is just a long hint to help you write a stand-alone proof from start to finish.)

Sketch of proof. For the remainder of the problem, fix $n \in \mathbf{N}$.

Step 1. Show that to prove that all norms on $\mathbf{R}^n$ are equivalent, it suffices to show that an arbitrary norm $\| \ \|$ on $\mathbf{R}^n$ is equivalent to the ℓ^1-norm $\| \ \|_1 := \| \ \|_{\ell^1}$ on $\mathbf{R}^n$. (*Any* fixed norm, not just the ℓ^1 norm, would serve this purpose; we simply choose a fixed norm that's easy to work with.) Then, for the remainder of the problem, fix an arbitrary norm $\| \ \|$ on $\mathbf{R}^n$.

Step 2. Let $S = \{y \in \mathbf{R}^n : \|y\|_1 = 1\}$. Show that S is closed and bounded in $(\mathbf{R}^n, \| \ \|_1)$. (Throughout, we use the usual convention for treating normed vector spaces as metric spaces: the implied metric is the one associated with the norm.)

Step 3. Let $\{e_i\}_{i=1}^n$ be the standard basis of $\mathbf{R}^n$ (the basis for which the i^{th} coordinate of e_i is 1 and all other coordinates are 0). Let $C = \max\{\|e_i\| : 1 \le i \le n\}$. Show that for all $x \in \mathbf{R}^n$, we have $\|x\| \le C\|x\|_1$.

Step 4. Using the result of Step 3, show that $g := \| \ \| : (\mathbf{R}^n, \| \ \|_1) \to \mathbf{R}$ defined by $g(x) = \| \ \|$ is continuous. (In other words, the norm-function $\| \ \|$ is continuous with respect to the metric defined by the *other* norm, $\| \ \|_1$.)

Step 5. Let $h = g|_S : S \to \mathbf{R}$. Applying earlier parts of this problem, show that h is a continuous function from a compact metric space to $\mathbf{R}$, and therefore (why?) achieves a minimum value m. Show also that $m > 0$.

Thus for all $y \in \mathbf{R}^n$ with $\|y\|_1 = 1$, we have $\|y\| \ge m > 0$.

Step 6. Use the result of Step 5 to show that for all $x \in \mathbf{R}^n$, we have $\|x\| \ge m\|x\|_1$.

Step 7. Combine the results of steps 6 and 3 to show that $\| \ \|$ is equivalent to $\| \ \|_1$.

Remark. As mentioned *but not proven* in class, for all $p \in [1, \infty)$ the function $\| \ \|_{\ell^p} : \mathbf{R}^n \to \mathbf{R}$ defined by $\|(x_1, \dots, x_n)\|_{\ell^p} = \left(\sum_{i=1}^n |x_i|^p\right)^{1/p}$ is a norm on $\mathbf{R}^n$. Modulo showing that each of these ℓ^p norms is, in fact, a norm, it is easy to show by explicit calculation that all these norms are equivalent; we do not need Theorem A3.1 for that. But there are many norms on $\mathbf{R}^n$ that are *not* ℓ^p norms for any $p \in [1, \infty]$.

B14. Prove the following corollary of Theorem A3.1.

Corollary A3.2: *Let V be a finite-dimensional vector space. Prove that all norms on V are equivalent (i.e. any two norms on V are equivalent to each other.)*

Hint: Use the fact that if $n = \dim(V) \in \mathbf{N}$ then V is isomorphic to $\mathbf{R}^n$. Choose an isomorphism $L : \mathbf{R}^n \to V$, and for every norm $\| \ \|$ on V, show that the function $\| \ \|_L : \mathbf{R}^n \to \mathbf{R}$ defined by $\|x\|_L = \|L(x)\|$ is a norm on $\mathbf{R}^n$. Then make appropriate use of Theorem A3.1.

Remark: In view of Corollary A3.2, all norms on a finite-dimensional vector space V determine the same open sets. This collection of open sets is called the *norm topology* on V.

B15. Prove another important, fundamental result:

Proposition A3.3: *Let $(V, \| \ \|_V), (W, \| \ \|_W)$ be normed vector spaces, and assume that V is finite-dimensional. Prove that every linear transformation $V \to W$ is continuous with respect to the given norms.*

Hint: Let $n = \dim(V)$. First deal with the case $n = 0$. Then assume $n \in \mathbf{N}$, choose a basis $\mathbf{e} := \{e_1, \ldots, e_n\}$ of V and let $\{x_i : V \to \mathbf{R}\}_{i=1}^n$ be the associated coordinate functions. (Recall that these are defined by $x_i(\sum_j a_j e_j) = a_i$; thus $v = \sum_j x_j(v) e_j$ for all $v \in V$.) Show that the map $\| \ \|_{1,\mathbf{e}} : V \to \mathbf{R}$ defined by $\|v\|_{1,\mathbf{e}} = \sum_{i=1}^n |x_i(v)|$ is a norm on V. Then find $C \geq 0$ such that $\|T(v)\|_W \leq C\|v\|_{1,\mathbf{e}}$. Use this, plus linearity, to show that $T : (V, \| \ \|_{1,\mathbf{e}} \to (W, \| \ \|_W)$ is Lipschitz, and therefore continuous. Then show that this implies that $T : (V, \| \ \|_V) \to (W, \| \ \|_W)$ is continuous.

Remark. An important corollary of Proposition A3.3 is the following:

Corollary A3.4: *Let $n \in \mathbf{N}$ and let $(V, \| \ \|)$ be an n-dimensional vector space. Let $\{e_i\}_{i=1}^n$ be a basis of V, and let $\{x_i\}_{i=1}^n$ be the corresponding coordinate functions. Then for each $i \in \{1, 2, \ldots, n\}$, the function $x_i : V \to \mathbf{R}$ is continuous.*

This follows from Proposition A3.3 and the fact that coordinate functions are linear transformations V from V to $\mathbf{R}$, hence are continuous.

Remark. If $V = \mathbf{R}^n$ and $\| \ \|$ is the ℓ^1, ℓ^2, or ℓ^∞ norm, then for $1 \leq i \leq n$ we have

$$\|a_i e_i\| = |a_i| \leq \|(a_1, \ldots, a_n)\|. \tag{2}$$

This makes it easy to prove directly (without results from this assignment) that the usual coordinate functions $\{x_i\}$ on $\mathbf{R}^n$ (the coordinate functions determined by the standard basis) are continuous with respect to these three norms. But there are norms on $\mathbf{R}^n$ for the inequality (2) is *not* true. Hence, without Theorem A3.1, it is not obvious that these coordinate functions are continuous with respect to *every* norm on $\mathbf{R}^n$. For the same reason, it is not trivial that the coordinate functions determined by an arbitrary basis of an arbitrary n-dimensional normed vector space are continuous.

B16. Let (Y, d_Y) be a metric space.

(a) Let X be a nonempty set. Prove that if (Y, d_Y) is complete, then so is $(B(X, Y), d_\infty)$, the space of bounded functions from X to Y.

(b) Let (X, d_X) be a metric space. Prove that if (Y, d_Y) is complete, then so is $(BC(X, Y), d_\infty)$, the space of bounded continuous functions from X to Y. (This completes the proof of Proposition 21.3 in the lecture notes.)

B17. (a) Show that the normed vector space $(B(\mathbf{N}, \mathbf{R}), \|\ \|_\infty)$ is precisely the space $(\ell^\infty(\mathbf{R}), \|\ \|_\infty)$ defined in Homework Assignment 1. (In this problem-part, "show" essentially means "check" or "observe". Part of what you're checking is that when our newer, more general definition of $\|\ \|_\infty$ on the space $B(X, \mathbf{R})$ for an arbitrary nonempty set X reduces to Assignment 1's definition of $\|\ \|_\infty$ on $\ell^\infty(\mathbf{R})$ if $X = \mathbf{N}$.) Thus, by problem B16(a), $\ell^\infty(\mathbf{R})$ is complete.

As noted in Assignment 1, the space $(\ell^\infty(\mathbf{R}), \|\ \|_\infty)$ is usually denoted just $\ell^\infty(\mathbf{R})$. We will do this below.

It is often useful to picture a sequence $(\vec{a}^{(m)})_{m=1}^\infty$ in $\ell^\infty(\mathbf{R})$ (or more generally in $\mathbf{R}^\infty$)—a sequence of sequences—as an array with infinitely many rows and columns, in which the first row is the sequence $\vec{a}^{(1)}$, the second row is the sequence $\vec{a}^{(2)}$, etc.:

$$\begin{array}{ccccc} a_1^{(1)} & a_2^{(1)} & a_3^{(1)} & a_4^{(1)} & \dots \\ a_1^{(2)} & a_2^{(2)} & a_3^{(2)} & a_4^{(2)} & \dots \\ a_1^{(3)} & a_2^{(3)} & a_3^{(3)} & a_4^{(3)} & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{array}$$

(b) Let $\vec{0}$ denote the zero element of $\ell^\infty(\mathbf{R})$, the sequence each of whose terms is $0 \in \mathbf{R}$. For $m \in \mathbf{N}$, let $\vec{e}^{(m)} \in \ell^\infty(\mathbf{R})$ be the sequence whose m^{th} term is 1 and all of whose other terms are zero (e.g. $\vec{e}^{(3)} = (0, 0, 1, 0, 0, 0, 0, \dots)$). Show that the sequence $E := (\vec{e}^{(m)})_{m=1}^\infty$ converges pointwise to $\vec{0}$.

Here, for the meaning of "pointwise", we are viewing $\ell^\infty(\mathbf{R})$ as $B(\mathbf{N}, \mathbf{R})$. Since the notation "$(x_i)_{i=1}^\infty$" for a real-valued sequence is simply convenient notation for the function $(i \in \mathbf{N}) \mapsto x_i \in \mathbf{R}$, saying that the sequence E converges pointwise to $\vec{0}$ is the same as saying that for each $i \in \mathbf{N}$, the real-valued sequence $(e_i^{(m)})_{m=1}^\infty$ converges to $0 \in \mathbf{R}$. In terms of the diagram above, this sequence is represented by the i^{th} *column* of the corresponding array.

(c) Use part (b) and results from class to prove that *if* the sequence E converges uniformly, then it must converge uniformly to $\vec{0}$.

(d) Compute $d_\infty(\vec{e}^{(m)}, \vec{0})$ for all n, and use your answer to show that E does not converge in $\ell^\infty(\mathbf{R})$ to $\vec{0}$. Hence, by part (c), E does not converge in $\ell^\infty(\mathbf{R})$, period.

(e) Compute $d(\vec{e}^{(n)}, \vec{e}^{(m)})$ for all $m, n \in \mathbf{N}$ with $m \neq n$. Use your answer to show that no subsequence of E can be Cauchy. Use this to deduce that no subsequence of E can converge.

(f) Use part (e) to deduce that the closed unit ball $\bar{B}_1(\vec{0}) \subseteq \ell^\infty(\mathbf{R})$ is not sequentially compact, hence is not compact.

Thus $\bar{B}_1(\vec{0})$ is a closed, bounded subset of a complete normed vector space, but is not compact. The Heine-Borel Theorem is false in infinite dimensions. More precisely, in the statement of the Heine-Borel Theorem, if we replace $(\mathbf{R}^n, \| \, \|)$ by an infinite-dimensional normed vector space, the statement we obtain is false.

B18. For any set X, let X^∞ denote the set of sequences in X. (Thus an element $\vec{x} \in X^\infty$ is a sequence $(x_n)_{n=1}^\infty$, where $x_n \in X$ for all $n \in \mathbf{N}$.) For a metric space (X, d), define a relation $\sim$ on X^∞ by

$$\vec{x} \sim \vec{w} \quad \text{if and only if} \quad \lim_{n \to \infty} d(x_n, w_n) = 0,$$

where $\vec{x} = (x_n)_{n=1}^\infty$ and $\vec{w} = (w_n)_{n=1}^\infty$. The same notation will be used below.

(a) Let (X, d) be a metrc space.

(i) Prove that $\sim$ is an equivalence relation on X^∞.

(ii) Let $\vec{x}, \vec{w} \in X^\infty$ and assume $\vec{x} \sim \vec{w}$. Prove that if $\vec{x}$ is Cauchy, then so is $\vec{w}$.

(iii) Let $\vec{x}, \vec{w} \in X^\infty$ and assume that $(x_n)_{n=1}^\infty$ converges to $p \in X$. Prove that $(w_n)_{n=1}^\infty$ converges to p if and only if $\vec{x} \sim \vec{w}$.

(b) Let (X, d_X) and (Y, d_Y) be metrc spaces, and let $f : X \to Y$ be a uniformly continuous function. Define $f_\infty : X^\infty \to Y^\infty$ by $f_\infty\big((x_n)_{n=1}^\infty \big) = \big(f(x_n) \big)_{n=1}^\infty$.

(i) Show that if $\vec{x}$ and $\vec{w}$ are sequences in X, and $\vec{x} \sim_X \vec{w}$, then $f_\infty(\vec{x}) \sim_Y f_\infty(\vec{w})$. (Since we have two different metric spaces X and Y, we've added subscripts to distinguish the two relevant equivalence relations.)

(ii) Show that if $\vec{x} \in X^\infty$ is Cauchy, then so is $f_\infty(\vec{x})$.

Assignment continues on next page.

B19. **Extensions of continuous functions with dense domains.** Recall that a subset S of a metric space X is called *dense* if the closure of S is the entire space X (e.g. $\mathbf{Q}$ is dense in $\mathbf{R}$); equivalently, if every point in $X \backslash S$ is a cluster point of S.

Let $(X, d_X), (Y, d_Y)$ be metric spaces, let S be a dense subset of X, and let $f : S \to Y$ be a function. Below we will assume that f is continuous ("just plain" or uniformly). In this problem we are interested in extensions of f to X—i.e. maps $\tilde{f} : X \to Y$ such that $\tilde{f}_S = f$—that have the same continuity property that f has. (This generalizes a problem considered last semester: if $f : (a, b) \to \mathbf{R}$ is a continuous function, does f extend continuously to $[a, b]$? We saw that in this setting, if f is *uniformly* continuous then the answer is yes.)

(a) Let $f : S \to Y$ be a continuous function, and suppose that $\tilde{f}_1, \tilde{f}_2$ are continuous extensions of f to X. Show that $\tilde{f}_1 = \tilde{f}_2$. (Thus a continuous extension of f to X, if any exists, is unique.)

(b) Assume that (Y, d_Y) is complete and that $f : S \to Y$ is *uniformly* continuous. Prove that f has a unique continuous extension to X, and that the extended function $\tilde{f}$ is uniformly continuous.

> *Hint*: In view of part (a), if we can establish existence of even a "just plain" continuous extension $\tilde{f}$, then $\tilde{f}$ will automatically be the unique such extension. Thus, the work will go into showing existence of a uniformly continuous extension $\tilde{f}$. To define an extension $\tilde{f}$, it suffices to define $\tilde{f}(x)$ for $x \in X \backslash S$. Since S is dense, for any such x there exists a sequence $\vec{x} := (x_n)_{n=1}^{\infty}$ converging to x. Using problem B18 and the completeness of (Y, d_Y), show that $\lim_{n\to\infty} f(x_n)$ exists and is independent of the choice of sequence $\vec{x}$ (i.e. if both $\vec{x}$ and $\vec{w}$ converge to x, then $\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} f(w_n)$). Use this fact to unambiguously define an appropriate element $\tilde{f}(x) \in Y$, and thereby to obtain an extension $\tilde{f} : X \to Y$. Finally, to conclude that $\tilde{f}$ is uniformly continuous, show that given any $x, w \in X$ and any sequences $\vec{x}, \vec{w} \in X^{\infty}$ converging to x, w respectively, $d_Y(\tilde{f}(x), \tilde{f}(w)) = \lim_{n\to\infty} d_Y(f(x_n), f(w_n))$, and then apply a fact you established during your proof of B18(b)(i).

Assignment continues on next page.

B20. **Length of a curve in a metric space.** A *curve* in a metric space (X, d) is a continuous function $f : [a, b] \to \mathbf{R}$ (where $[a, b]$ can be any closed, bounded, positive-length interval in $\mathbf{R}$). The *length* of a curve $f : [a, b] \to X$ is defined to be

$$\ell(f) := \sup \left\{ \sum_{i=1}^{N} d(f(x_{i-1}, f(x_i)) : \{x_0, x_1, \ldots, x_N\} \text{ is a partition of } [a, b] \right\} \tag{3}$$

provided that the supremum exists, i.e. that the set of sums in (3) is bounded. (When this supremum exists, we say that the length of the curve exists, or is finite, or that the curve *has length*; otherwise we say that the length of the curve does not exist, or is infinite, or that the curve does not have length.) "Partition of $[a, b]$" in (3) has the same meaning as in Riemann integration, but the sums in (3) are, in general, *not* Riemann sums of any function $g : [a, b] \to \mathbf{R}$.

When $X = \mathbf{R}^n$, a curve f can be written in the form $(f_1, f_2, \ldots, f_n)$, where f_i s a real-valued function on $[a, b]$, $1 \le i \le n$. (Alternatively, $f = f_1 \oplus f_2 \oplus \ldots \oplus f_n$), where we generalize our definiton of the direct sum of two functions from $[a, b] \to \mathbf{R}$ to the direct sum of n such functions.) We say that f is *continuously differentiable* if each of the component functions f_i is continuously differentiable.

Prove that if (X, d) is Euclidean space $\mathbf{E}^n := (\mathbf{R}^n, d_{\text{Euc}}) = (\mathbf{R}^n, d_{\ell^2})$, and $f : [a, b] \to \mathbf{R}$ is a continuously differentiable curve, then the length of the curve f exists and is equal to

$$\int_a^b \sqrt{f_1'(t)^2 + f_2'(t)^2 + \cdots + f_n'(t)^2}\, dt.$$

(You will probably find this the most difficult problem I have assigned. Here are some hints to reduce the difficulty: (1) the Mean Value Theorem is relevant, but **does not generalize to vector-valued functions**. (2) Something that the MVT, if correctly used, will lead you to write down, is *not* a Riemann sum, but can be related to a Riemann sum by applying "a continuous real-valued function on a compact set is uniformly continuous" to the right function and compact set.

If you have what you think is a quick proof that doesn't involve the MVT and uniform continuity, you are probably overlooking something, making an implicit assumption, etc.) mm-mmmm