MAA 4212, Spring 2021—Assignment 4's non-book problems

Problems B1–B5 are the same as Assignment 3's problems B16–B20, except that the (current) B5(d) has reworded to give you two ways of reaching that problem-part's conclusion.

B1. Let (Y, d_Y) be a metric space.

(a) Let X be a nonempty set. Prove that if (Y, d_Y) is complete, then so is $(B(X, Y), d_\infty)$, the space of bounded functions from X to Y.

(b) Let (X, d_X) be a metric space. Prove that if (Y, d_Y) is complete, then so is $(BC(X, Y), d_\infty)$, the space of bounded continuous functions from X to Y. (This completes the proof of Proposition 21.3 in the lecture notes.)

B2. (a) Show that the normed vector space $(B(\mathbf{N}, \mathbf{R}), \| \ \|_\infty)$ is precisely the space $(\ell^\infty(\mathbf{R}), \| \ \|_\infty)$ defined in Homework Assignment 1. (In this problem-part, "show" essentially means "check" or "observe". Part of what you're checking is that when our newer, more general definition of $\| \ \|_\infty$ on the space $B(X, \mathbf{R})$ for an arbitrary nonempty set X reduces to Assignment 1's definition of $\| \ \|_\infty$ on $\ell^\infty(\mathbf{R})$ if $X = \mathbf{N}$.) Thus, by problem B16(a), $\ell^\infty(\mathbf{R})$ is complete.

As noted in Assignment 1, the space $(\ell^\infty(\mathbf{R}), \| \ \|_\infty)$ is usually denoted just $\ell^\infty(\mathbf{R})$. We will do this below.

It is often useful to picture a sequence $(\vec{a}^{(m)})_{m=1}^\infty$ in $\ell^\infty(\mathbf{R})$ (or more generally in $\mathbf{R}^\infty$)—a sequence of sequences—as an array with infinitely many rows and columns, in which the first row is the sequence $\vec{a}^{(1)}$, the second row is the sequence $\vec{a}^{(2)}$, etc.:

$$\begin{array}{ccccc} a_1^{(1)} & a_2^{(1)} & a_3^{(1)} & a_4^{(1)} & \dots \\ a_1^{(2)} & a_2^{(2)} & a_3^{(2)} & a_4^{(2)} & \dots \\ a_1^{(3)} & a_2^{(3)} & a_3^{(3)} & a_4^{(3)} & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{array}$$

(b) Let $\vec{0}$ denote the zero element of $\ell^\infty(\mathbf{R})$, the sequence each of whose terms is $0 \in \mathbf{R}$. For $m \in \mathbf{N}$, let $\vec{e}^{(m)} \in \ell^\infty(\mathbf{R})$ be the sequence whose m^{th} term is 1 and all of whose other terms are zero (e.g. $\vec{e}^{(3)} = (0, 0, 1, 0, 0, 0, 0, \dots)$). Show that the sequence $E := (\vec{e}^{(m)})_{m=1}^\infty$ converges pointwise to $\vec{0}$.

Here, for the meaning of "pointwise", we are viewing $\ell^\infty(\mathbf{R})$ as $B(\mathbf{N}, \mathbf{R})$. Since the notation "$(x_i)_{i=1}^\infty$" for a real-valued sequence is simply convenient notation for the function $(i \in \mathbf{N}) \mapsto x_i \in \mathbf{R}$, saying that the sequence E converges pointwise to $\vec{0}$ is the same as saying that for each $i \in \mathbf{N}$, the real-valued sequence $(e_i^{(m)})_{m=1}^\infty$ converges to $0 \in \mathbf{R}$. In terms of the diagram above, this sequence is represented by the i^{th} *column* of the corresponding array.

(c) Use part (b) and results from class to prove that *if* the sequence E converges uniformly, then it must converge uniformly to $\vec{0}$.

(d) Compute $d_\infty(\vec{e}^{(m)}, \vec{0})$ for all n, and use your answer to show that E does not converge in $\ell^\infty(\mathbf{R})$ to $\vec{0}$. Hence, by part (c), E does not converge in $\ell^\infty(\mathbf{R})$, period.

(e) Show, in either of the following two ways (your choice) that no subsequence of E can converge in $\ell^\infty(\mathbf{R})$:

(i) Compute $d(\vec{e}^{(n)}, \vec{e}^{(m)})$ for all $m, n \in \mathbf{N}$ with $m \neq n$. Use your answer to show that no subsequence of E can be Cauchy, and hence that no subsequence of E can converge in $\ell^\infty(\mathbf{R})$,

(ii) Show that every subsequence of E converges pointwise to $\vec{0}$, use your calculation in part (d) to show that no subsequence of E converges uniformly to $\vec{0}$, and then use the same argument as and then use an argument similar to part (c)'s to show that no subsequence of E can converge in $\ell^\infty(\mathbf{R})$.

(f) Use part (e) to deduce that the closed unit ball $\bar{B}_1(\vec{0}) \subseteq \ell^\infty(\mathbf{R})$ is not sequentially compact, hence is not compact.

Thus $\bar{B}_1(\vec{0})$ is a closed, bounded subset of a complete normed vector space, but is not compact. The Heine-Borel Theorem is false in infinite dimensions. More precisely, in the statement of the Heine-Borel Theorem, if we replace $(\mathbf{R}^n, \| \, \|)$ by an infinite-dimensional normed vector space, the statement we obtain is false.

B3. For any set X, let X^∞ denote the set of sequences in X. (Thus an element $\vec{x} \in X^\infty$ is a sequence $(x_n)_{n=1}^\infty$, where $x_n \in X$ for all $n \in \mathbf{N}$.) For a metric space (X, d), define a relation $\sim$ on X^∞ by

$$\vec{x} \sim \vec{w} \text{ if and only if } \lim_{n\to\infty} d(x_n, w_n) = 0,$$

where $\vec{x} = (x_n)_{n=1}^\infty$ and $\vec{w} = (w_n)_{n=1}^\infty$. The same notation will be used below.

(a) Let (X, d) be a metric space.

(i) Prove that $\sim$ is an equivalence relation on X^∞.

(ii) Let $\vec{x}, \vec{w} \in X^\infty$ and assume $\vec{x} \sim \vec{w}$. Prove that if $\vec{x}$ is Cauchy, then so is $\vec{w}$.

(iii) Let $\vec{x}, \vec{w} \in X^\infty$ and assume that $(x_n)_{n=1}^\infty$ converges to $p \in X$. Prove that $(w_n)_{n=1}^\infty$ converges to p if and only if $\vec{x} \sim \vec{w}$.

(b) Let (X, d_X) and (Y, d_Y) be metric spaces, and let $f : X \to Y$ be a uniformly continuous function. Define $f_\infty : X^\infty \to Y^\infty$ by $f_\infty\big((x_n)_{n=1}^\infty \big) = \big(f(x_n) \big)_{n=1}^\infty$.

(i) Show that if $\vec{x}$ and $\vec{w}$ are sequences in X, and $\vec{x} \sim_X \vec{w}$, then $f_\infty(\vec{x}) \sim_Y f_\infty(\vec{w})$. (Since we have two different metric spaces X and Y, we've added subscripts to distinguish the two relevant equivalence relations.)

(ii) Show that if $\vec{x} \in X^\infty$ is Cauchy, then so is $f_\infty(\vec{x})$.

B4. **Extensions of continuous functions with dense domains.** Recall that a subset S of a metric space X is called *dense* if the closure of S is the entire space X (e.g. $\mathbf{Q}$ is dense in $\mathbf{R}$); equivalently, if every point in $X \setminus S$ is a cluster point of S.

Let $(X, d_X), (Y, d_Y)$ be metric spaces, let S be a dense subset of X, and let $f : S \to Y$ be a function. Below we will assume that f is continuous ("just plain" or uniformly). In this problem we are interested in extensions of f to X—i.e. maps $\tilde{f} : X \to Y$ such that $\tilde{f}_S = f$—that have the same continuity property that f has. (This generalizes a problem considered last semester: if $f : (a, b) \to \mathbf{R}$ is a continuous function, does f extend continuously to $[a, b]$? We saw that in this setting, if f is *uniformly* continuous then the answer is yes.)

(a) Let $f : S \to Y$ be a continuous function, and suppose that $\tilde{f}_1, \tilde{f}_2$ are continuous extensions of f to X. Show that $\tilde{f}_1 = \tilde{f}_2$. (Thus a continuous extension of f to X, if any exists, is unique.)

(b) Assume that (Y, d_Y) is complete and that $f : S \to Y$ is *uniformly* continuous. Prove that f has a unique continuous extension to X, and that the extended function $\tilde{f}$ is uniformly continuous.

> *Hint*: In view of part (a), if we can establish existence of even a "just plain" continuous extension $\tilde{f}$, then $\tilde{f}$ will automatically be the unique such extension. Thus, the work will go into showing existence of a uniformly continuous extension $\tilde{f}$. To define an extension $\tilde{f}$, it suffices to define $\tilde{f}(x)$ for $x \in X \setminus S$. Since S is dense, for any such x there exists a sequence $\vec{x} := (x_n)_{n=1}^\infty$ converging to x. Using problem B3 and the completeness of (Y, d_Y), show that $\lim_{n\to\infty} f(x_n)$ exists and is independent of the choice of sequence $\vec{x}$ (i.e. if both $\vec{x}$ and $\vec{w}$ converge to x, then $\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} f(w_n)$). Use this fact to unambiguously define an appropriate element $\tilde{f}(x) \in Y$, and thereby to obtain an extension $\tilde{f} : X \to Y$. Finally, to conclude that $\tilde{f}$ is uniformly continuous, show that given any $x, w \in X$ and any sequences $\vec{x}, \vec{w} \in X^\infty$ converging to x, w respectively, $d_Y(\tilde{f}(x), \tilde{f}(w)) = \lim_{n\to\infty} d_Y(f(x_n), f(w_n))$, and then apply a fact you established during your proof of B3(b)(i).

B5. **Length of a curve in a metric space.** A *curve* in a metric space (X, d) is a continuous function $f : [a, b] \to \mathbf{R}$ (where $[a, b]$ can be any closed, bounded, positive-length interval in $\mathbf{R}$). The *length* of a curve $f : [a, b] \to X$ is defined to be

$$\ell(f) := \sup \left\{ \sum_{i=1}^{N} d(f(x_{i-1}), f(x_i)) : \{x_0, x_1, \ldots, x_N\} \text{ is a partition of } [a, b] \right\} \tag{1}$$

provided that the supremum exists, i.e. that the set of sums in (1) is bounded. (When this supremum exists, we say that the length of the curve exists, or is finite, or that the curve *has length*; otherwise we say that the length of the curve does not exist, or is infinite, or that the curve does not have length.) "Partition of $[a, b]$" in (1) has the same meaning as in Riemann integration, but the sums in (1) are, in tgeneral, *not* Riemann sums of any function $g : [a, b] \to \mathbf{R}$.

When $X = \mathbf{R}^n$, a curve f can be written in the form $(f_1, f_2, \ldots, f_n)$, where f_i s a real-valued function on $[a, b]$, $1 \le i \le n$. (Alternatively, $f = f_1 \oplus f_2 \oplus \ldots \oplus f_n$), where we generalize our definiton of the direct sum of two functions from $[a, b] \to \mathbf{R}$ to the direct sum of n such functions.) We say that f is *continuously differentiable* if each of the component functions f_i is continuously differentiable.

Prove that if (X, d) is Euclidean space $\mathbf{E}^n := (\mathbf{R}^n, d_{\text{Euc}}) = (\mathbf{R}^n, d_{\ell^2})$, and $f : [a, b] \to \mathbf{R}^n$ is a continuously differentiable curve, then the length of the curve f exists and is equal to

$$\int_a^b \sqrt{f_1'(t)^2 + f_2'(t)^2 + \cdots + f_n'(t)^2}\, dt.$$

(You will probably find this the most difficult problem I have assigned. Here are some hints to reduce the difficulty: (1) the Mean Value Theorem is relevant, but **does not generalize to vector-valued functions**. (2) Something that the MVT, if correctly used, will lead you to write down, is *not* a Riemann sum, but can be related to a Riemann sum by applying "a continuous real-valued function on a compact set is uniformly continuous" to the right function and compact set.

If you have what you think is a quick proof that doesn't involve the MVT and uniform continuity, you are probably overlooking something, making an implicit assumption, etc.)

B6. Define $\tilde{f} : ([0, \infty) \times [0, \infty)) \setminus \{(0, 0)\} \to \mathbf{R}$ by $\tilde{f}(x, y) = x^y$. Let f be the restriction of $\tilde{f}$ to the domain $(0, \infty) \times [0, \infty)$; observe that $\text{domain}(\tilde{f}) = \text{domain}(f) \coprod (\{0\} \times (0, \infty)) = \text{domain}(f) \coprod (\text{positive } y\text{-axis})$.

Below, every subset of $U \subseteq \mathbf{R}^2$ is given the metric induced by a chosen norm on $\mathbf{R}^2$. As shown earlier homework, all norms on $\mathbf{R}^2$ are equivalent (hence yield equivalent metrics), and equivalent metrics on U determine the same continuous functions from U to $\mathbf{R}$. Thus the truth of the assertions you are asked to prove below does not depend on which norm you choose on $\mathbf{R}^2$, so you may choose any norm on $\mathbf{R}^2$ that you find convenient.

(a) Show that $\tilde{f}$ is the unique continuous extension of f to $([0,\infty)\times[0,\infty))\setminus\{(0,0)\}$. (Note that you need to show two facts about $\tilde{f}$, not necessarily in the following order: (i) that $\tilde{f}$ is continuous, and (ii) that any continuous extension of f to $([0,\infty)\times[0,\infty))\setminus\{(0,0)\}$ is the function $\tilde{f}$.)

(b) Show that $\lim_{x\to 0}(\lim_{y\to 0} f(x,y))$ and $\lim_{y\to 0}(\lim_{x\to 0} f(x,y))$ exist but are not equal.

(c) Show that there does not exist a continuous extension of f to $[0,\infty)\times[0,\infty) = \text{domain}(\tilde{f})\cup\{(0,0)\}$. (This is why we do not define "0^0".) *Suggestion*: Use the fact that any such extension would also be an extension of $\tilde{f}$.

(d) Define $\tilde{g} : ([0,\infty)\times\mathbf{R})\setminus(\{0\}\times(-\infty,0]) \to \mathbf{R}$ by $\tilde{g}(x,y) = x^y$. Let g be the restriction of $\tilde{g}$ to the domain $(0,\infty)\times\mathbf{R}$; observe that $\text{domain}(\tilde{g}) = \text{domain}(g)\coprod(\{0\}\times(0,\infty))$. Redo parts (a) and (c) with $\tilde{f}$ and f replaced by $\tilde{g}$ and g, respectively.

B7. Let $\vec{a} := (a_n)_{n=1}^{\infty}$ be a sequence in $\mathbf{R}$ for which $\sum_{n=1}^{\infty} a_n$ is conditionally convergent (i.e. convergent but not absolutely convergent). Prove the following:

(a) There are infinitely many n for which a_n is positive, and infinitely many n for which a_n is negative.

(b) Let $(b_i)_{i=1}^{\infty}$ be the subsequence of $\vec{a}$ consisting of the positive terms of $\vec{a}$. (I.e. $b_i = a_{n_i}$, where n_i is the index of the i^{th} positive term of $\vec{a}$.) Similarly let $(c_i)_{i=1}^{\infty}$ be the subsequence of $\vec{a}$ consisting of the negative terms of $\vec{a}$. Show that both of the series $\sum_{i=1}^{\infty} b_i$, $\sum_{i=1}^{\infty} c_i$ diverge.

(c) Let r be any element of the extended reals $\mathbf{R}_{\text{ext}}$. Prove that there exists a rearrangement of $\sum_{n=1}^{\infty} a_n$ that converges in $\mathbf{R}_{\text{ext}}$ to r. In other words, prove that (i) for any *real* number r, there exists a bijection $f:\mathbf{N}\to\mathbf{N}$ such that $\sum_{n=1}^{\infty} a_{f(n)} = r$; (ii) there exists a bijection $f:\mathbf{N}\to\mathbf{N}$ such that $\sum_{n=1}^{\infty} a_{f(n)}$ diverges to ∞ (i.e. which *con*verges in $\mathbf{R}_{\text{ext}}$ to ∞), and (iii) there exists a bijection $f:\mathbf{N}\to\mathbf{N}$ such that $\sum_{n=1}^{\infty} a_{f(n)}$ diverges to $-\infty$.

B8. Let $(a_{(m,n)})_{(m,n)\in\mathbf{N}\times\mathbf{N}}$ be a "doubly indexed sequence" in $\mathbf{R}$—a map $A:\mathbf{N}\times\mathbf{N}\to\mathbf{R}$, where $a_{(m,n)} = A(m,n)$. It is sometimes useful to picture $(a_{(m,n)})$ as an "infinity-by-infinity matrix". In this problem we are interested in attaching meaning to the notation "$\sum_{m,n\in\mathbf{N}\times\mathbf{N}} a_{(m,n)}$", also written $\sum_{m,n} a_{(m,n)}$ or $\sum_{m,n=1}^{\infty} a_{(m,n)}$. Our notation "$a_{(m,n)}$" is sometimes just written "$a_{m,n}$".

Definition. The doubly-indexed series $\sum_{m,n} a_{(m,n)}$ is *absolutely convergent* (or *converges absolutely*) if there exists a bijection $f:\mathbf{N}\to\mathbf{N}\times\mathbf{N}$ such that $\sum_{j=1}^{\infty} a_{f(j)}$ is absolutely convergent. (Said more loosely, we are calling the doubly-indexed series absolutely convergent if there is some order in which we can add up the entries of the "infinite matrix" $(a_{(m,n)})$ as the terms of an absolutely convergent singly-indexed series.)

(a) Prove that if $\sum_{m,n} a_{(m,n)}$ converges absolutely and $f, g : \mathbf{N} \to \mathbf{N} \times \mathbf{N}$ are bijections, then $\sum_{j=1}^{\infty} a_{f(j)} = \sum_{j=1}^{\infty} a_{g(j)}$. Hence if $\sum_{m,n} a_{(m,n)}$ converges absolutely, we can unambiguously define

$$\sum_{m,n} a_{(m,n)} = \sum_{j=1}^{\infty} a_{f(j)}$$

where f is *any* bijection $\mathbf{N} \to \mathbf{N} \times \mathbf{N}$.

(b) Explain why we should not attach any numerical value (in $\mathbf{R}$) to the notation "$\sum_{m,n} a_{(m,n)}$" if this doubly-indexed series is *not* absolutely convergent. (*Hint*: Problem B7(c).)

(c) Prove that if $\sum_{m,n} a_{(m,n)}$ is absolutely convergent then (i) for all $m \in \mathbf{N}$, the series $\sum_{n=1}^{\infty} a_{(m,n)}$ converges, (ii) for all $n \in \mathbf{N}$, the series $\sum_{m=1}^{\infty} a_{(m,n)}$ converges, and (iii)

$$\sum_{m,n} a_{(m,n)} = \sum_{m=1}^{\infty} \left(\sum_{n=1}^{\infty} a_{(m,n)} \right) = \sum_{n=1}^{\infty} \left(\sum_{m=1}^{\infty} a_{(m,n)} \right).$$

(d) Let $\sum_{n=1}^{\infty} b_n$, $\sum_{n=1}^{\infty} c_n$ be absolutely convergent series in $\mathbf{R}$. Prove that $\sum_{m,n} b_m c_n$ is absolutely convergent, and that

$$\sum_{m,n} b_m c_n = \left(\sum_{n=1}^{\infty} b_n \right) \left(\sum_{n=1}^{\infty} c_n \right).$$

Remark. In the absolutely convergent case, enumerating $\mathbf{N} \times \mathbf{N}$ in the order

$$\begin{array}{c} (1,1), \\ (1,2),\ (2,1), \\ (1,3),\ (2,2),\ (3,1), \\ \cdots \end{array}$$

(i.e. the first term of the enumeration is the one on the first line; the next two terms are the ones on the second line, from left to right; the next three terms are the ones on the third line, from left to right; etc.) leads us to

$$\sum_{m,n \in \mathbf{N}} a_{(m,n)} = \sum_{k=1}^{\infty} \left(\sum_{m+n=k} a_{(m,n)} \right). \tag{2}$$

(The notation above for the inner sum on the right-hand side of (2) is understood to mean the sum of all terms whose index-pair (m, n) lies in $\mathbf{N} \times \mathbf{N}$ and satisfies $m + n = k$.

We could also write this inner sum as $\sum_{m=1}^{k-1} a_{(m,k-m)}$, or as $\sum_{n=1}^{k-1} a_{(k-n,n)}$, or with other dummy variables for the index of summation: $\sum_{j=1}^{k-1} a_{(j,k-j)}$ etc.) One of the main reasons that the conclusions of problem B8 are important is the following application to power series, in which the enumeration scheme in (2) appears naturally. (For power series, we index the terms using $\mathbf{N} \cup \{0\}$ rather than $\mathbf{N}$, but aside from the slight bookkeeping change, which includes starting the outer sum in (2) at $k = 0 = 0 + 0$ instead of at $k = 2 = 1 + 1$, this clearly makes no difference in the conclusions of B8.)

Suppose you are multiplying two polynomials together, say $a_0 + a_1 x + \cdots + a_N x^N$ (i.e. $\sum_{n=0}^{N} a_n x^n$) and $b_0 + b_1 x + \cdots + b_M x^M$ (i.e. $\sum_{m=0}^{M} b_m x^m$). After multiplying out, you generally rewrite the result by grouping together all the terms with a given power of x, which is the finite-series statement

$$\left(\sum_{n=0}^{N} a_n x^n\right)\left(\sum_{m=0}^{M} b_m x^m\right) = \sum_{k=0}^{N+M}\left(\sum_{n+m=k} a_n b_m\right) x^k.$$

We can extend this *fact* about products of polynomials to a *definition* of the formal product of two formal power series: the formal product of $\sum_{n=0}^{\infty} a_n x^n$ and $\sum_{n=0}^{\infty} b_n x^n$ is defined to be $\sum_{k=0}^{\infty}\left(\sum_{n+m=k} a_n b_m\right) x^k$, multiplying the formal power series as if the they were "polynomials with infinitely many terms". Note that this definition does not require any convergence; it amounts to no more than defining a sequence of coefficients $(c_k = \sum_{n+m=k} a_n b_m)_{k=0}^{\infty}$. But since power series are absolutely convergent on the their open intervals of convergence, parts (a) and (d) imply that on the smaller of the open intervals of convergence of the power series $\sum_n a_n x^n$ and $\sum_n b_n x^n$, the formal product converges to the product of the *functions* $F : x \mapsto \sum_{n=0}^{\infty} a_n x^n$ and $G : x \mapsto \sum_{n=0}^{\infty} b_n x^n$. Hence, in this case, the formal product is the unique power-series representation, centered at 0 and with variable x, of the product function FG.

9. Let $\sum_{n=0}^{\infty} b_n (x-a)^n$ and $\sum_{n=0}^{\infty} c_n (x-a)^n$ be power series with positive radii of convergence R_1 and R_2, respectively, and let f and g, respectively, be the analytic functions represented by these power series on the corresponding open intervals of convegence.

(a) Show that $f+g$ is analytic at a, and that its power-series representation centered at a Taylor series centered at a has radius of convergence $R \geq \min\{R_1, R_2\}$.

(b) Repeat part (a), but for the prodiuct fg.

10. Last semester we proved that $e^x e^y = e^{x+y}$ for all $x, y \in \mathbf{R}$. Give another proof of this fact using power series and problem B8(d).