MAA 4212, Spring 2021—Assignment 5's non-book problems
(possibly not complete yet)

B1. Exercise 1.1 in the Multivariable Derivatives handout ("Some Notes on Multivariable Derivatives")

B2. Exercise 1.2 in the Multivariable Derivatives handout.

B3. Exercise 1.3 in the Multivariable Derivatives handout.

B4. Exercise 1.4 in the Multivariable Derivatives handout.

B5. Exercise 4.1 in the Multivariable Derivatives handout.

B6. Exercise 4.2 in the Multivariable Derivatives handout.

B7. Exercise 6.1 in the Multivariable Derivatives handout.

B8. (**Product Rule as a corollary of Chain Rule.**)

(a) Define $\mu : \mathbf{R}^2 \to \mathbf{R}$ by $\mu(\begin{pmatrix} x \\ y \end{pmatrix}) = xy$. (In case you don't know: "μ" is the lower-case Greek letter "mu".) Compute the Jacobian matrix of μ at an arbitrary point $p \in \mathbf{R}^2$ ($J_\mu(p)$ in my notation; $D\mu(p)$ in Tao's notation).

(b) Let $U \subset \mathbf{R}$ be open, let $f, g : U \to \mathbf{R}$ be differentiable, and define $F : U \to \mathbf{R}^2$ by $F(t) = \begin{pmatrix} f(t) \\ g(t) \end{pmatrix}$. As proven in class, F is differentiable since its component functions f, g are differentiable. Write down the Jacobian matrix of F at an arbitrary point $t \in U$. (This is just the "$m = 2$" case of Example 4.3 in the Multivariable Derivatives handout. You're just rewriting the answer here for use in part (c).)

(c) With all data as in (a) and (b), define $h : U \to \mathbf{R}$ by $h = \mu \circ F$. Apply the Chain Rule (what I called the "second best" version, the one involving Jacobians) to compute the Jacobian of h.

You should find that you have just proved the Product Rule from Calc 1 as a corollary of the (multivariable) Chain Rule. This brings me to:

Groisser's philosophy of calculus (circa 1980):

"Calculus consists of three theorems:

–the Chain Rule Theorem;
–the Inverse/Implicit Function Theorem;
–Stokes's Theorem.

The rest is corollary."

(In this philosophy, the first line is prefaced by an implicit "Once you really understand it", and all of single-variable calculus is considered a special case of multivariable calculus.)

The Inverse Function Theorem and Implicit Function Theorem count as just one theorem on my list, because they're (non-obviously) equivalent to each other. The Stokes Theorem on my list is a much more general theorem than the one called by that name in Calc 3; it has all ot the following as special cases: the Fundamental Theorem of Calculus; the Fundamental Theorem of Line Integrals; Green's Theorem in the plane, the Divergence Theorem; and the Calc-3 version of Stokes's Theorem.

Over time I added a fourth theorem to the list—the change-of-variables formula for multiple integrals (which you saw for double and triple integrals in Calc 3, without a proof). The Mean Value Theorem and Equality of Cross-Partials (Clairaut's Theorem) have migrated onto and off the list a few times.

I've also found that the final sentence of my philosophy was too pithy to be accurate. More accurate would be "The rest is corollary, definition, or lemma needed to prove these theorems," but, alas, that doesn't have the same ring as the original.

Calculus is something I *love*. My philosophy isn't that the items omitted from my list are unimportant or trivial; it's that the items *on* my list are **so** important **to calculus, proper**, that the rest is just not in the same tier.

B9. Let $(V, \langle\ ,\ \rangle)$ be finite-dimensional inner-product space (a finite-dimensional vector space V equipped with an inner product $\langle\ ,\ \rangle$).[1] Let $\|\ \|$ be the norm associated with the inner product ($\|v\| = \sqrt{\langle v, v, \rangle}$).

(a) Define $f : V \to \mathbf{R}$ by $f(x) = \langle x, x\rangle = \|x\|^2$. (i) For all $x, v \in V$, compute the directional derivative $(D_v f)(x)$. (ii) Show that f is differentiable, and for all $x \in V$ compute the derivative $df|_x \in \text{Hom}(V, \mathbf{R})$ for all $x \in V$. (Since $df|_x$ is a linear map from V to $\mathbf{R}$, "computing" $df|_x$ amounts to computing $df|_x(v)$ for all $v \in \mathbf{R}$. By the time you've shown that f is differentiable, you'll already have done this.)

[1] LaTeX tip: for the angle-brackets in "$\langle\ ,\ \rangle$", don't use less-than/greater-than symbols; they have the wrong look and are confusing to read. Use \langle and \rangle instead.

(b) Define $g : V \to \mathbf{R}$ by $g(x) = \langle x, x \rangle = \|x\| = \sqrt{f(x)}$. Show that g is differentiable on $V \backslash \{0_V\}$, and give a formula for all the directional derivatives $(D_v g)(x)$ $(x \in V \backslash \{0_V\}, \quad v \in V)$.

B10. Let f_1 and $f_\infty : \mathbf{R}^2 \to \mathbf{R}$ denote the ℓ^1 and ℓ^∞ norms.

(a) For each of these functions f, determine the set of points in $\mathbf{R}^2$ at which f is differentiable. (*Note*: You would find analogous answers if $\mathbf{R}^2$ were replaced by $\mathbf{R}^n$. I'm just saving you some writing.)

(b) From problem B1, we already know that the ℓ^2-norm is differentiable except at the origin. You should find that the ℓ^1 and ℓ^∞ norms have more points of non-differentiability than the ℓ^2-norm has. But for a given function $g : \mathbf{R}^2 \to \mathbf{R}$, the differentiability of g at a given point does not depend on the choice of norm on $\mathbf{R}^2$ (as you were asked to prove, on the previous homework assignment, for any function from (open set $U \subseteq (V, \| \ \|_V)$) to $(W, \| \ \|_W)$. Why isn't this a contradiction?