Non-book problems for Assignment 1

NB 1.1. In class, I stated two versions of the Axiom of Induction (AoI), involving the following conditions on a subset S of $\mathbf{N}$, the set of positive integers (natural numbers):

(i) $1 \in S$.

(ii) For any $n \in \mathbf{N}$, if $n \in S$ then $n+1 \in S$.

(ii)$'$ For any $n \in \mathbf{N}$, if every natural number $j \leq n$ lies in S, then $n+1 \in S$.

The first version of the AoI that I stated was: if $S \subseteq \mathbf{N}$ and S satisfies conditions (i) and (ii), then $S = \mathbf{N}$. The second version I stated was: if $S \subseteq \mathbf{N}$ and S satisfies conditions (i) and (ii)$'$, then $S = \mathbf{N}$. In class I stated why the first version implies the second, and gave an informal argument why the second version implies the first (and thus that the two versions are equivalent). Write out a *careful* proof of each of these implications.

The terms "first version" and "second version" of the AoI are just names I've chosen for convenience. I'll continue to use them below, but they aren't mathematically "official" terminology.

NB 1.2 Gallian's Theorems 0.5 and 0.6 state two "Principles of Mathematical Induction" (a "First Principle" and a "Second Principle") for subsets of $\mathbf{Z}$, the set of *all* integers, not just positive integers. (*Note:* Each of Theorem 0.5 and 0.6 is missing what should have been its first sentence, "Let a be an integer.") The first and second versions of the AoI are the $a = 1$ cases of these two principles; thus Gallian's "first and second principles" of induction imply the first and second versions, respectively, of the AoI. Show, conversely, that each version of the AoI implies Gallian's corresponding "principle of mathematical induction", and hence that each version of the AoI is *equivalent* to Gallian's corresponding principle.

(As you probably realized, Gallian's "a" in these principles corresponds to what you're probably used to calling the "base case" of an inductive argument. Note that a is allowed to be *any* integer—positive, negative, or zero.)

NB 1.3

(a) Show that the Well-Ordering Principle (WOP) implies the first version of the Axiom of Induction. I.e., assume the WOP, and deduce the (first version) of the AoI.

(Suggestion: Given a set $S \subseteq \mathbf{N}$ satisfying the conditions (i) and (ii) in problem NB 1.1, consider the *complement* of S in $\mathbf{N}$, the set $S' := \{n \in \mathbf{N} \mid n \notin S\}$.)

(b) Show that the second version of the AoI implies the WOP. (Suggestion: Given a nonempty subset $T \subseteq \mathbf{N}$, assume that T does *not* have a smallest element, and see if you can apply the second version of the AoI to the complement of T in $\mathbf{N}$.)

Once you are done with problems NB 1.1, 1.2, and 1.3: calling the two versions of the Axiom of Induction "AoI1" and "AoI2" (temporarily), and calling Gallian's First and Second Principles of Mathematical Induction "PMI1" and "PMI2" (temporarily), you have now shown the following implications:

$$\text{WOP} \implies \text{AoI 1}, \quad (1)$$
$$\text{AoI2} \implies \text{WOP}, \quad (2)$$
$$\text{AoI1} \iff \text{AoI2}, \quad (3)$$
$$\text{AoI1} \iff \text{PMI1}, \quad (4)$$
$$\text{and} \quad \text{AoI2} \iff \text{PMI2}. \quad (5)$$

Thus, *all five* of WOP, AoI1, AoI2, PMI1, and PMI2 are equivalent (each implies all of the others). For example, AoI1 $\implies$ AoI2 $\implies$ WOP (using (3) and (2)) and PMI2 $\implies$ AoI2 $\implies$ AoI1 $\implies$ PMI1 (using (5), (3), and (4)).