Non-book problems for Assignment 2

NB 2.1. Let G be a group, let H be a subgroup of G, and let K be a subgroup of H. Show that K is a subgroup of G. (Thus, said informally, a "sub-subgroup" is a subgroup.)

> Using the "$\leq$" notation for the subgroup relation, what this exercise asserts is: "If $K \leq H$ and $H \leq G$, then $K \leq G$;" i.e. the subgroup relation "$\leq$" is transitive. But remember that **there is no such thing as "proof by notation"!!**

NB 2.2. Let G and G' be two finite groups, of equal order n, and suppose that we have listed the elements of G and G':

$$G = \{g_1, g_2, \ldots, g_n\}, \quad G' = \{g'_1, g'_2, \ldots, g'_n\}.$$

We consider G and G' to be "essentially the same", and say that they are *isomorphic* to each other, if they have the same multiplication table (i.e. if, for all $i, j, k \in \{1, 2, \ldots, n\}$, we have that $g'_i g'_j = g'_k$ iff $g_i g_j = g_k$), or do so after reordering the elements of one of the groups. (We'll define "isomorphic" more generally than for finite groups when we get to Chapter 6.) For example, if G_1 and G_2 are *cyclic* groups of order n, they are "essentially the same" in this sense.[1]

Your mission: find a non-cyclic group of order 4, and show that any two such groups are "essentially the same", but are not "essentially the same" as $\mathbf{Z}_4$.

Hint to get you started: A non-cyclic group of order 4 must have at least two non-identity elements a and b that are not powers of each other, and must have exactly one other element besides e, a, and b. Construct a multiplication table (Cayley table) consistent with this information, and show that your table is the *only* such table. Note that in a multiplication table for a group, each element of the group must appear exactly once in each column and in each row (why?).

NB 2.3. Find all the subgroups of $\mathbf{Z}$, and prove that your answer is correct. (There are infinitely many subgroups, but they can all be listed systematically.) What does this tell you about the subgroups of every other infinite cyclic group?

NB 2.4. Show that the infinite abelian groups $\mathbf{R}$ and $\mathbf{Q}$ (with addition as the group operation in both cases) are *not* cyclic.

[1] In a week or two, instead of saying "essentially the same", we'll say "isomorphic"