Non-book problems for Assignment 4

NB 4.1. Let $n \in \mathbf{Z}_+$, let A be a set of cardinality n, and let $\mathrm{Perm}(A)$ denote the group of permutations of A (the set of all bijective functions from $A \to A$, with composition as the group operation). The goal of this problem is to prove that $\mathrm{Perm}(A)$ is isomorphic to S_n. The first step is to come up with a *candidate* for an isomorphism, a map from one group to the other that we hope has the right properties.

Since A has cardinality n, we can list its elements as (say) $a_1, a_2, \ldots, a_n$. Note that in doing so, we have chosen a bijection f from the set $\{1, 2, \ldots, n\}$ to A; the "i^{th} element" a_i of A is simply $f(i)$, $1 \le i \le n$. Given a permutation π of A, we seek a permutation π' of $\{1, \ldots, n\}$ that corresponds to π if the one-to-one correspondence f is used to translate between "$\{1, \ldots, n\}$-language" and "A-language", meaning that acting on an element $i \in \{1, \ldots, n\}$ by the permutation $\pi' \in S_n$, then mapping the result $\pi'(i)$ to A via f, should yield the same result as acting on $f(i) \in A$ by the permutation π of A. Letting $a_i = f(i)$, what we therefore want to have is: $\pi(a_i) = a_{\pi'(i)}$ for each $i \in \{1, \ldots n\}$. Writing this just in terms of f (without the "a_i" notation), what we want is to have $\pi(f(i)) = f(\pi'(i))$. Thus, what we want is for π and π' to be related by $\pi \circ f = f \circ \pi'$; equivalently (since f is a bijection), $\pi' = f^{-1} \circ \pi \circ f$. This suggests our *candidate* for an isomorphism from $\mathrm{Perm}(A)$ to A_n: the function $\phi : \mathrm{Perm}(A) \to S_n$ defined by

$$\phi(\pi) = f^{-1} \circ \pi \circ f.$$

Show that ϕ is, in fact, an isomorphism from $\mathrm{Perm}(A)$ to S_n.

NB 4.2. Let $n \ge 2$ be an integer. For each $k \in \{1, \ldots, n\}$, the *stabilizer* of k in S_n is

$$\mathrm{stab}_{S_n}(k) := \{\alpha \in S_n : \alpha(k) = k\}.$$

A special case of the result you showed in Gallian's exercise Ch. 5/ 35 is that, for any $k \in \{1, \ldots, n\}$ the stabilizer of k (in S_n) is a subgroup of S_n.

(a) Observe that the elements of $\mathrm{stab}_{S_n}(n)$ are exactly those permutations of S_n that permute the elements of $\{1, \ldots,\ n-1\}$ amongst themselves: if $\alpha \in \mathrm{stab}_{S_n}(n)$ and $i \in \{1, \ldots,\ n-1\}$, then $\alpha(i)$ cannot be n, since $\alpha(n) = n$ and α is one-to-one. Thus, in the two-row-array notation for α,

$$\alpha = \begin{bmatrix} 1 & 2 & \dots & n-1 & n \\ \alpha(1) & \alpha(2) & \dots & \alpha(n-1) & n \end{bmatrix}$$
$$= \begin{bmatrix} 1 & 2 & \dots & n-1 & n \\ \alpha'(1) & \alpha'(2) & \dots & \alpha'(n-1) & n \end{bmatrix},$$

where $\alpha' \in S_{n-1}$ is the permutation defined by $\alpha'(i) = \alpha(i),\ 1 \leq i \leq n-1$. Show that the map $\phi : \text{stab}_{S_n}(n) \to S_{n-1}$ is an isomorphism, and hence that $\text{stab}_{S_n}(n)$ is isomorphic to S_{n-1}.

(b) For $k < n$, the elements of $\text{stab}_{S_n}(k)$ can similarly be identified with permutations of the $(n-1)$-element set $\{1, 2, \dots, n\} \setminus \{k\}$, so we might intuitively expect that $\text{stab}_{S_n}(k)$ is again isomorphic to S_{n-1}. But the first step in showing this—coming up with a *candidate* for an isomorphism—is not as simple as it was for the case $k = n$, since $\{1, 2, \dots, n\} \setminus \{k\} \neq \{1, 2, \dots, n\} \setminus \{n\} = \{1, 2, \dots, n-1\}$. If you try to write down an explicit candidate for an isomorphism from $\text{stab}_{S_n}(k)$ (where $k < n$), you will find the task more challenging than when $k = n$. Instead, use the result of problem NB 4.1 to help you show that $\text{stab}_{S_n}(k)$ is, indeed, isomorphic to S_{n-1}.

(c) Using part (b), show that for any $j, k \in \{1, 2, \dots, n\}$, the groups $\text{stab}_{S_n}(j)$ and $\text{stab}_{S_n}(k)$ are isomorphic.

(d) There is another way to prove the result of part (c) that is much quicker, but is not based on the same intuition. Without your having seen an example of this approach before, it may appear to be like pulling a rabbit out of a hat.

The idea is this: given distinct $j, k \in S_n$, perhaps we can find an inner automorphism $\tilde{\phi}$ of S_n that carries $\text{stab}_{S_n}(j)$ to $\text{stab}_{S_n}(k)$. If there is such an automorphism ϕ, then by Ch. 6/ 34—as modified on homework page so that it would be useful here—the subgroups $\text{stab}_{S_n}(j)$ and $\text{stab}_{S_n}(k)$ are isomorphic.

The only automorphisms that we know an arbitrary group has are inner automorphisms, so we'll look for an inner automorphism that might do we want. The only obvious diffference between $\text{stab}_{S_n}(j)$ and $\text{stab}_{S_n}(k)$ is that the roles of j and k. So perhaps a permutation in S_n that interchanges the roles of j and k. could be helpful.

Now for your part of the job: Find the simplest permutation β you can think of in S_n that interchanges the roles of j and k, and show that the inner automorphism ϕ_β (the map $g \mapsto \beta g \beta^{-1}$) does, indeed, carry $\text{stab}_{S_n}(j)$ to $\text{stab}_{S_n}(k)$, thereby establishing that the two subgroups are isomorphic to each other.