Non-book problems for Assignment 5

NB 5.1. Let G be a finite group each of whose non-identity elements has order 2.

(a) Prove that G is Abelian.

(b) Prove that the order of G is a power of 2 (i.e. $|G| = 2^m$ for some integer $m \geq 0$).

> *Hint for part (b).* If G has a subgroup H of order 2^k, and H is not all of G, then there is some $a \in G$ that is not in H. Show that $H \cup aH$ is a subgroup of G of order 2^{k+1}. (Part (a) will enter this argument.)
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> Note: "Hint" does not mean "Show what the hint says, and then just assert that the result of the problem follows." This particular hint has its own hypothesis—one that's not assumed explicitly in the problem statement—and the conclusion of the hint does not say anything about what the order of G *equals*.

NB 5.2. (a) Let G_1 and G_2 be groups and assume that $f : G_1 \to G_2$ is an isomorphism (thus, we are implicitly assuming that G_1 and G_2 are isomorphic). Let α be an automorphism of G_1. Show that $f \circ \alpha \circ f^{-1}$ is an automorphism of G_2.

(b) Let G_1 and G_2 be isomorphic groups. Prove that $\mathrm{Aut}(G_1) \approx \mathrm{Aut}(G_2)$.

[**Note that since G_1, G_2, f, and α were introduced within part (a), and part (a) ended before part (b) began—writing "(b)" tells us that part (a) is over—the "Let" statement and any notation introduced in part (a) had expired by the time we wrote the first word of part (b); at that time, there was nothing currently being called G_1, G_2, f, or α. If problem 5.2's setup had said "Let G_1 and G_2 be groups" at the very beginning, *before* part (a), then each of G_1 and G_2 would have been a fixed (but arbitrary) group for all parts of the problem. But because this notation was introduced within part (a), we could not use it in part (b) without re-introducing it. (Also, if we wanted to use this notation now to mean something different from what it meant in part (a), we'd be free to do that.)** I wrote the two-part problem the way I did so that part (b) could be a free-standing problem. Part (b) was originally going to be the whole problem; I decided to insert part (a) to help you with part (b).]

If you notice some formal similarity between problem 5.2 above and NB problem 4.1, you are not going crazy. However, NB problem 4.1 can't be used to do NB problem 5.2 or vice-versa; there is simply a theme—an important, general one—that is common to both.

NB 5.3. Let G_1 and G_2 be groups. Show that $G_1 \oplus G_2 \approx G_2 \oplus G_1$.