Non-book problems for Assignment 6

NB 6.1. (This problem was already done verbally in class.) In the dihedral group D_4, let H be the subgroup consisting of the four rotations, let g be any reflection, and let $K = \langle g \rangle$. Since H is a cyclic group of order 4, and and group of order 2 is cyclic, $H \approx \mathbf{Z}_4$ and $K \approx \mathbf{Z}_2$.

(a) Without computing conjugates of any element, why do we know automatically that $H \triangleleft D_4$?

(b) Show that $D_4 = HK$.

(c) Without attempting to decide directly whether the subgroup K is normal, how can we see quickly that D_4 is *not* isomorphic to $\mathbf{Z}_4 \oplus \mathbf{Z}_2$, hence that D_4 cannot be the internal direct product $H \times K$. (Is there a simple property that one of these 8-element groups $\mathbf{Z}_4 \oplus \mathbf{Z}_2$ has that the other does not, and that would be preserved by any isomorphism between $\mathbf{Z}_4 \oplus \mathbf{Z}_2$ and D_4?)

(d) Recall that a corollary of Theorem 9.7 is that for any prime p, every group of order p^2 is Abelian. Give an example showing that this would be false if we replaced p^2 by p^3.