Non-book problems for Assignment 8

NB 8.1. Let R be any ring, not necessarily commutative, and let $M_{2\times 2}(R)$ be the set of 2×2 matrices with entries in R. We define addition and multiplication operations on $M_{2\times 2}(R)$ the same way as if R were $\mathbf{R}$ or $\mathbf{Z}$ or $\mathbf{Z}_2$:

$$\begin{aligned} \begin{pmatrix} a & b \\ c & d \end{pmatrix} + \begin{pmatrix} f & g \\ h & j \end{pmatrix} &= \begin{pmatrix} a+f & b+g \\ c+h & d+j \end{pmatrix}, \\ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} f & g \\ h & j \end{pmatrix} &= \begin{pmatrix} af+bh & ag+bj \\ cf+dh & cg+dj \end{pmatrix}, \end{aligned}$$

where $a,b,c,d,f,g,h,j \in R$. The only difference is that, if R is noncommutative, we have to be careful that we write a product such as af in the indicated order (and not to change it in any subsequent calculations), with the first term coming from the first matrix and the second term coming from the second matrix.

(a) Check that $M_{2\times 2}(R)$, equipped with these operations, is a ring. (Here and below, I'm saying "check" rather than "show", because we've done almost all of this whole problem in class.)

(b) Check that if R is finite, so is $M_{2\times 2}(R)$.

(c) Check that if R has an identity element (a unity), then so does $M_{2\times 2}(R)$.

(d) Check that even if R is commutative, $M_{2\times 2}(R)$ need not be commutative.

Thus, even if we start with familiar, commutative rings like $R = \mathbf{Z}$ or $R = \mathbf{Z}_n$, we can construct noncommutative rings. We can then use these to give examples of rings of 2×2 matrices in which the entries come from a commutative ring; e.g. we can form the ring $M_{2\times 2}(M_{2\times 2}(\mathbf{Z}))$.

NB 8.2. Formulate a definition of *ring homomorphism* and *ring isomorphism.* Write it down.

Check your answer by looking at the first page of Chapter 15, but **do the formulating first and the checking second!** The point of this problem is to exercise your *brain*, not your ability to turn pages or search/scroll through a document. I want you to understand the *concept* of "homomorphism" and "isomorphism" for groups well enough to figure out what these words *ought* to mean for rings.