Differential forms on $\mathbf{R}^3$
Notes for MAT 4930—Curves and Surfaces—Spring 2019

In these notes, elements of $\mathbf{R}^3$ are denoted in boldface; elements of general vector spaces are not.

1 Multi-covectors

Definition 1.1 Let V, Z be vector spaces, k a positive integer, and

$$B: \overbrace{V \times V \times \cdots \times V}^{k\text{-fold Cartesian product}} \to Z \tag{1}$$

a function. (On your first reading, to make the ideas as concrete as possible, mentally replace Z by $\mathbf{R}$. Other Z's will not enter till the next section of these notes.) We call B *multilinear* if for $1 \leq i \leq k$, when all variables $v_1, v_2, \ldots, v_k$ in the expression $B(v_1, v_2, \ldots, v_k)$ are held fixed except the i^{th}, the resulting function of the i^{th} variable is a linear map from V to Z.

For $k = 1$, a multilinear function is just a linear map. For $k = 2$, we also call a multilinear function bilinear; for $k = 3$, a multilinear function is also called trilinear, etc. When k is not given a specific value, the term "k-linear" is also used.

Example 1.2 With notation as above, a function $B : V \times V \to Z$ is bilinear (or multilinear) if for all $w \in V$, the two maps from V to Z given by

$$\begin{aligned} v &\mapsto B(v, w) \\ \text{and} \quad v &\mapsto B(w, v) \end{aligned}$$

are linear. A function $B : V \times V \times V \to Z$ is trilinear (or multilinear) if for all $w_1, w_2 \in V$, the three maps from V to Z given by

$$\begin{aligned} v &\mapsto B(v, w_1, w_2), \\ v &\mapsto B(w_1, v, w_2), \\ \text{and} \quad v &\mapsto B(w_1, w_2, v) \end{aligned}$$

are linear.

Definition 1.3 Notation as in Definition 1.1. A bilinear function $B : V \times V \to Z$ is called *antisymmetric* if

$$B(v, w) = -B(w, v) \tag{2}$$

for all $v, w \in V$. For $k \geq 2$, a multilinear function is called *alternating* (or *antisymmetric*, or *totally antisymmetric*) if whenever all but two of the variables in the expression

$B(v_1, v_2, \ldots, v_k)$ are held fixed, the resulting function of the remaining two variables is antisymmetric.

It is convenient not to have to say "assume $k \geq 2$" when referring to alternating k-linear functions, so we call linear maps from V to W (the case $k = 1$) alternating as well, even though there aren't two variables to interchange.

Remark 1.4 Note that the antisymmetry condition in equation (2) implies that $B(v, v) = 0$ for all $v \in V$. Similarly, for any $k > 2$, if B is k-linear and alternating, and $v_1, v_2, \ldots v_k \in V$, and there are two indices $i \neq j$ for which $v_i = v_j$, then $B(v_1, v_2, \ldots, v_k) = 0$.

Example 1.5 An alternating trilinear map $B : V \times V \times V \to \mathbf{R}$ obeys the conditions

$$\begin{aligned} B(v_1, v_2, v_3) &= -B(v_2, v_1, v_3) \quad \text{(swapping the first and second vectors)}, \\ B(v_1, v_2, v_3) &= -B(v_3, v_2, v_1) \quad \text{(swapping the first and third vectors)}, \\ B(v_1, v_2, v_3) &= -B(v_1, v_3, v_2) \quad \text{(swapping the second and third vectors)}, \end{aligned}$$

for all $v_1, v_2, v_3 \in V$. Using these relations, it is easy to show that if σ is a permutation of $\{1, 2, 3\}$, then $B(v_{\sigma(1)}, v_{\sigma(2)}, v_{\sigma(3)}) = \pm B(v_1, v_2, v_3)$, where the sign is the sign of the permutation σ. (If you don't know what "sign of a permutation" means, just ignore the last statement.) This generalizes to k-linear alternating multilinear functions $V \times V \times \cdots \times V \to Z$ for any vector space Z and any $k > 0$.

Remark 1.6 For any nonempty set S, the set $\mathrm{Func}(S, \mathbf{R})$ of *all* functions from S to $\mathbf{R}$ is a vector space. (The notation "$\mathrm{Func}(S, \mathbf{R})$" is just for these notes, not universal.) The vector-space operations are the usual operations on functions: for $f, g \in \mathrm{Func}(S, \mathbf{R})$ and $c \in \mathbf{R}$, the functions $f + g \in \mathrm{Func}(S, \mathbf{R})$ and $cf \in \mathrm{Func}(S, \mathbf{R})$ by

$$\begin{aligned} (f + g)(s) &= f(s) + g(s) \quad \text{for all } s \in S, \\ (cf)(s) &= c\, f(s) \quad \text{for all } s \in S. \end{aligned}$$

As the student may check, for any vector space V and positive integer k, the set of all k-linear functions from $V \times V \times \cdots \times V$ to $\mathbf{R}$ is closed under addition and under multiplication by scalars, and thus is a subspace of $\mathrm{Func}(V \times V \times \cdots \times V, \mathbf{R})$. For this reason, we often use the terminology "the *space* of k-linear functions from $V \times V \times \cdots \times V$ to $\mathbf{R}$" for this set. Similarly, we speak of the *space* of alternating k-linear functions from $V \times V \times \cdots \times V$ to $\mathbf{R}$, because the set of such functions is closed under the above operations, and is thus a subspace of the space of all k-linear functions from $V \times V \times \cdots \times V$ to $\mathbf{R}$. All of the preceding (in this Remark) remains true if the codomain $\mathbf{R}$ is replaced by a general vector space.

Suppose now that V has (finite) dimension $n > 0$, and let $\{u_1, \ldots, u_n\}$ be a basis. Suppose B is a k-linear alternating function from $V \times V \times \cdots \times V$ to $\mathbf{R}$. Given any vectors $v_1, \ldots v_k \in V$, we can express each v_i as a linear combination of the given basis vectors, and then use the multilinearity of B to express $B(v_1, v_2, \ldots, v_k)$ as a linear combination of expressions of the form $B(u_{i_1}, u_{i_2}, \ldots, u_{i_k})$, where $(i_1, \ldots, i_k)$ ranges over all ordered k-tuples of integers in the set $\{1, 2, \ldots, n\}$. If $k > n$, then for any such k-tuple at least two of the indices i_j must be equal, so two of the vectors u_{i_j} must be equal, so by Remark 1.4, $B(u_{i_1}, u_{i_2}, \ldots, u_{i_k}) = 0$. Since a linear combination of 0's is 0, this implies that $B(v_1, v_2, \ldots, v_k) = 0$. But the v_i were *arbitrary* vectors in V. Hence

$$\begin{array}{l}\text{If } k > \dim(V)\text{, then the only alternating } k\text{-linear}\\ \text{function from } V \times \cdots \times V \text{ to } \mathbf{R} \text{ is the zero function.}\end{array} \tag{3}$$

Definition 1.7 (For these notes; not universal) Let $k \geq 0$ be an integer and let $\mathbf{p} \in \mathbf{R}^3$. If $k > 0$, a *k-covector at* $\mathbf{p}$ is an alternating multilinear function

$$\phi_{\mathbf{p}} : \overbrace{T_{\mathbf{p}}\mathbf{R}^3 \times T_{\mathbf{p}}\mathbf{R}^3 \times \cdots \times T_{\mathbf{p}}\mathbf{R}^3}^{k\text{-fold Cartesian product}} \to \mathbf{R}.$$

(Thus a 1-covector at $\mathbf{p}$ is just a covector at $\mathbf{p}$.) We define "0-covector at $\mathbf{p}$" to mean "real number". We will write $\mathcal{A}^k_{\mathbf{p}}$ for the space of all k-covectors at $\mathbf{p}$; the word "space" here is a reminder that (by Remark 1.6 in the case $k > 0$) the set of all k-covectors at $\mathbf{p}$ is a vector space. In the notation "$\mathcal{A}^k_{\mathbf{p}}$", the k is just a superscript, not an exponent; don't say "to the k".

Note that by our definition of "0-covector", $\mathcal{A}^0_{\mathbf{p}} = \mathbf{R}$.

A k-covector is said to have *degree* k.

If $\phi_{\mathbf{p}}$ is a k-covector at $\mathbf{p}$, and $k > 3$, then by statement (3) we have $\phi_{\mathbf{p}} = 0$, where this "0" is the zero-element of the vector space $\mathcal{A}^k_{\mathbf{p}}$. Thus:

$$\text{For } k > 3\text{, the space } \mathcal{A}^k_{\mathbf{p}} \text{ is a zero vector space.} \tag{4}$$

(A "zero vector space", also called a "trivial vector space", is a vector space whose only element is its zero element. We often refer to any such vector space as "the" zero vector space, since there is an obvious and unique bijection between any two sets having just one element each.)

For terminological simplicity, we allow the term "multi-covector" to mean any element of any of the spaces $\mathcal{A}^k_{\mathbf{p}}$ (even if $k = 0$ or $k = 1$).

2 Wedge product at a point

For each $\mathbf{p} \in \mathbf{R}^3$ and all integers $k, l \geq 0$, we define a binary operation

$$\begin{array}{rcl} \wedge : \mathcal{A}^k_{\mathbf{p}} \times \mathcal{A}^l_{\mathbf{p}} & \to & \mathcal{A}^{k+l}_{\mathbf{p}}, \\ (\phi_{\mathbf{p}}, \psi_{\mathbf{p}}) & \mapsto & \phi_{\mathbf{p}} \wedge \psi_{\mathbf{p}}, \end{array} \tag{5}$$

called *wedge product*, as follows. To make the formulas below easier to read, we omit the subscript $\mathbf{p}$ from tangent vectors and (multi-)covectors.

1. For $c \in \mathcal{A}^0_{\mathbf{p}}$ and $\psi \in \mathcal{A}^l_{\mathbf{p}}$, we define $c \wedge \psi$ to be $c\psi \in \mathcal{A}^l_{\mathbf{p}} = \mathcal{A}^{k+l}_{\mathbf{p}}$. Similarly, if $l = 0$ then for $\phi \in \mathcal{A}^k_{\mathbf{p}}$ and $c \in \mathcal{A}^l_{\mathbf{p}} = \mathcal{A}^0_{\mathbf{p}}$, we define $\phi \wedge c$ to be $c\phi \in \mathcal{A}^k_{\mathbf{p}} = \mathcal{A}^{k+l}_{\mathbf{p}}$.

2. For $\phi, \psi \in \mathcal{A}^1_{\mathbf{p}}$ we define $\phi \wedge \psi : T_{\mathbf{p}}\mathbf{R}^3 \times T_{\mathbf{p}}\mathbf{R}^3 \to \mathbf{R}$ by

$$\phi \wedge \psi \ (\mathbf{v}, \mathbf{w}) = \phi(\mathbf{v})\psi(\mathbf{w}) - \phi(\mathbf{w})\psi(\mathbf{v}) \tag{6}$$

   The student should check that, with this definition, $\phi \wedge \psi$ is indeed a bilinear, alternating function from $T_{\mathbf{p}}\mathbf{R}^3 \times T_{\mathbf{p}}\mathbf{R}^3$ to $\mathbf{R}$, i.e. an element of $\mathcal{A}^2_{\mathbf{p}} = \mathcal{A}^{1+1}_{\mathbf{p}}$.

3. For $\phi \in \mathcal{A}^1_{\mathbf{p}}$ and $\psi \in \mathcal{A}^2_{\mathbf{p}}$ we define $\phi \wedge \psi : T_{\mathbf{p}}\mathbf{R}^3 \times T_{\mathbf{p}}\mathbf{R}^3 \times T_{\mathbf{p}}\mathbf{R}^3 \to \mathbf{R}$ by

$$\phi \wedge \psi \ (\mathbf{u}, \mathbf{v}, \mathbf{w}) = \phi(\mathbf{u})\psi(\mathbf{v}, \mathbf{w}) + \phi(\mathbf{v})\psi(\mathbf{w}, \mathbf{u}) + \phi(\mathbf{w})\phi(\mathbf{u}, \mathbf{v}) \tag{7}$$

   (Observe that on the right-hand side of (7), we have a sum over *cyclic* permutations of $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$. We would have obtained the same value had we summed over all permutations, analogously to (6), and divided the answer by 2.) The student should check that, with this definition, $\phi \wedge \psi$ is indeed a trilinear, alternating function from $T_{\mathbf{p}}\mathbf{R}^3 \times T_{\mathbf{p}}\mathbf{R}^3 \times T_{\mathbf{p}}\mathbf{R}^3$ to $\mathbf{R}$, i.e. an element of $\mathcal{A}^3_{\mathbf{p}} = \mathcal{A}^{1+2}_{\mathbf{p}}$.

4. For $\phi \in \mathcal{A}^2_{\mathbf{p}}$ and $\psi \in \mathcal{A}^1_{\mathbf{p}}$ we define $\phi \wedge \psi$ to be $\psi \wedge \phi$, which we have just defined above.

5. For $\phi \in \mathcal{A}^k_{\mathbf{p}}$ and $\psi \in \mathcal{A}^l_{\mathbf{p}}$ in all cases not handled above, we have $k + l > 3$, so $\mathcal{A}^{k+l}_{\mathbf{p}}$ is a zero vector space and we define $\phi \wedge \psi$ to be the zero element of this space.

Note that what we have actually defined is a *collection* of maps that we could have labeled "$\wedge_{k,l}$" according to the degrees of multi-covectors being wedged together. These labeled maps will be referred to once below, in a footnote. However, labeling them wherever "$\wedge$" appears would lead to getting lost in a forest of notation.

With wedge-product now defined, the student may check that the following properties hold:

1. For all $k, l \geq 0$, the map (5) is bilinear. In particular, the left-distributive and right-distributive laws hold.

2. Wedge product is associative in the following sense[1]: for all $k, l, m \geq 0$, and all $\phi \in \mathcal{A}^k_{\mathbf{p}}, \psi \in \mathcal{A}^l_{\mathbf{p}}, \mu \in \mathcal{A}^m_{\mathbf{p}}$, we have

$$(\phi \wedge \psi) \wedge \mu = \phi \wedge (\psi \wedge \mu). \tag{8}$$

[1]The words "in the following sense" have been added since, in equation (8), up to four different wedge-product maps are involved: reading from left to right, the first $\wedge$ is the one we could have labeled $\wedge_{k,l}$, the second is the one we could have labeled $\wedge_{k+l,m}$, the third is the one we could have labeled $\wedge_{k,l+m}$, and the fourth is the one we could have labeled $\wedge_{l,m}$.

3. If $\phi \in \mathcal{A}^k_{\mathbf{p}}$ and $\psi \in \mathcal{A}^l_{\mathbf{p}}$ then

$$\psi \wedge \phi = (-1)^{kl} \phi \wedge \psi. \tag{9}$$

(This can be stated by saying: under wedge product, even-degree covectors commute with all covectors, and odd-degree covectors *anti*commute with each other.) Note that for $k+l \leq 3$, the only case in which $(-1)^{kl}$ is negative is the case $k = l = 1$; in all other cases this sign is positive. If $k+l > 3$ then both sides of equation (9) are zero, so the equality is trivial.

Observe that to check these properties there is just a small set of pairs (k,l) or triples (k,l,m) for which one needs to check. For example, both sides of equation (8) are automatically zero if $k+l+m > 3$, so one need only check the case $k = l = m = 1$ and the cases in which (at least) one of k,l,m is zero. The latter cases follow from the definition of "wedge with a 0-covector" and the bilinearity of the map (5) (once one has established bilinearity).

3 Differential forms

As defined in class, a 1-form is a *covector field*: a function $\phi : \mathbf{R}^3 \to \bigcup_{\mathbf{p}\in\mathbf{R}^3} \mathcal{A}^1_{\mathbf{p}}$, which whose value at $\mathbf{p}$ we chose to denote $\phi_{\mathbf{p}}$ rather than $\phi(\mathbf{p})$, satisfying the requirement that $\phi_{\mathbf{p}}$ lie in $\mathcal{A}^1_{\mathbf{p}}$ and satisfying a certain smoothness condition. A general k-form on $\mathbf{R}^3$ is defined by just replacing the superscript 1 with k (and specifying the smoothness condition, which we will do later). In other words:

Definition 3.1 For $k \geq 0$, a *k-form* on $\mathbf{R}^3$ is a (smooth) k-covector field, where a *k-covector field* is a map $\phi : \mathbf{R}^3 \to \bigcup_{\mathbf{p}\in\mathbf{R}^3} \mathcal{A}^k_{\mathbf{p}}$ such that $\phi_{\mathbf{p}} := \phi(\mathbf{p}) \in \mathcal{A}^k_{\mathbf{p}}$ for each $\mathbf{p} \in \mathbf{R}^3$.

A k-form is also called a *differential form of degree k*.

For $k \geq 0$, let $\tilde{\mathcal{A}}^k$ denote the (vector) space of all (not necessarily smooth) k-covector fields on $\mathbf{R}^3$. (The simpler notation $\mathcal{A}^k$, without the tilde, is being held in reserve for the space of smooth k-covector fields.) Temporarily, let us ignore the eventual smoothness condition and refer to elements of $\tilde{\mathcal{A}}^k$ as k-forms. With this understood, for all integers $k,l \geq 0$ we define a binary operation

$$\begin{aligned} \wedge : \tilde{\mathcal{A}}^k \times \tilde{\mathcal{A}}^l &\to \tilde{\mathcal{A}}^{k+l}, \\ (\phi,\psi) &\mapsto \phi \wedge \psi, \end{aligned}$$

again called *wedge product*, using the pointwise-principle: for every $\mathbf{p} \in \mathbf{R}^3$, we define

$$(\phi \wedge \psi)_{\mathbf{p}} = \phi_{\mathbf{p}} \wedge \psi_{\mathbf{p}} \in \tilde{\mathcal{A}}^{k+l}_{\mathbf{p}}.$$

It is easily seen that the algebraic properties of wedge-product of covectors (bilinearity, associativity, and the "signed-commutativity" property (9)) are inherited by wedge-product of differential forms. In fact, the bilinearity property can be upgraded to "$\mathcal{F}$-bilinearity": $(f\phi+g\psi)\wedge\mu = f\phi\wedge\mu+g\psi\wedge\mu$ and $\phi\wedge(f\psi+g\mu) = f\phi\wedge\psi+g\phi\wedge\mu$, where $f, g : \mathbf{R}^3 \to \mathbf{R}$ are arbitrary functions (that's what the "$\mathcal{F}$" stands for), not just real numbers.

The differentials of the coordinate functions x, y, z provide us with an easy way of writing down a basis of each of the spaces $\mathcal{A}^k_{\mathbf{p}}$ for $\mathbf{p} \in \mathbf{R}^3$ and $1 \le k \le 3$, and thereby understand the spaces $\tilde{\mathcal{A}}^k$. Writing $dx_{\mathbf{p}}, dy_{\mathbf{p}}, dz_{\mathbf{p}}$ for the covectors at $\mathbf{p}$ given by these differentials[2], it follows from classwork that that $\{dx_{\mathbf{p}}, dy_{\mathbf{p}}, dz_{\mathbf{p}}\}$ is a basis of $\mathcal{A}^1_{\mathbf{p}}$ (in particular, this space is three-dimensional). Using definitions and facts established earlier in these notes, it is then not hard to show that $\{dx_{\mathbf{p}} \wedge dy_{\mathbf{p}},\ dx_{\mathbf{p}} \wedge dz_{\mathbf{p}},\ dy_{\mathbf{p}} \wedge dz_{\mathbf{p}}\}$ is a basis of $\mathcal{A}^2_{\mathbf{p}}$ (thus this space is also three-dimensional) and that the singleton-set $\{dx_{\mathbf{p}} \wedge dy_{\mathbf{p}} \wedge dz_{\mathbf{p}}\}$ is a basis of $\mathcal{A}^3_{\mathbf{p}}$ (a space that is, therefore, one-dimensional).[3] Using the pointwise-principle, it follows that

- every zero-form on $\mathbf{R}^3$ is a function $f : \mathbf{R}^3 \to \mathbf{R}$, and conversely every such function is a 0-form;
- every 1-form on $\mathbf{R}^3$ can be written uniquely as $f\,dx + g\,dy + h\,dz$ for some functions $f, g, h : \mathbf{R}^3 \to \mathbf{R}$, and conversely every such expression is a 1-form;
- every 2-form on $\mathbf{R}^3$ can be written uniquely as $f\,dx \wedge dy + g\,dx \wedge dz + h\,dy \wedge dz$ for some functions $f, g, h : \mathbf{R}^3 \to \mathbf{R}$, and conversely every such expression is a 2-form;
- every 3-form on $\mathbf{R}^3$ can be written uniquely as $f\,dx \wedge dy \wedge dz$ for some function $f : \mathbf{R}^3 \to \mathbf{R}$, and conversely every such expression is a 3-form.

We have already defined what "smooth" (C^∞) means for a function $f : \mathbf{R}^3 \to \mathbf{R}$. We now define a 1-form $f\,dx + g\,dy + h\,dz$ to be smooth if the coefficient functions f, g, h are smooth; define a 2-form $f\,dx \wedge dy + g\,dx \wedge dz + h\,dy \wedge dz$ to be smooth if the coefficient functions f, g, h are smooth; and define a 3-form $f\,dx \wedge dy \wedge dz$ to be smooth if the coefficient function f is smooth.

For $k \ge 0$, let $\mathcal{A}^k \subset \tilde{\mathcal{A}}^k$ denote the set of *smooth* k-forms.

Given two differential forms $\phi \in \mathcal{A}^k, \psi \in \mathcal{A}^l$ with $k + l \le 3$, using $\mathcal{F}$-bilinearity and the "signed commutativity" property it is easy to express the coefficient function(s) of $\phi \wedge \psi$ as a sum and/or difference of (ordinary) products of the coefficient-functions of ϕ and ψ. Since sums, differences, and products of smooth real-valued functions are smooth, it follows that the coefficient function(s) of $\phi \wedge \psi$ is/are smooth. Hence $\phi \wedge \psi$ lies in $\mathcal{A}^{k+l}$ (not just $\tilde{\mathcal{A}}^{k+l}$). Thus, the set of smooth differential forms is closed under wedge-product.

Finally, we redefine "k-form" and "differential form" to mean what have have called "smooth k-form" and "smooth differential form" above.

[2]With this notation one must remember that $\mathbf{p}$ is a subscript here for the expressions dx, dy, dz, not for x, y, z. Notation such as $dx|_{\mathbf{p}}$ or $(dx)_{\mathbf{p}}$ (etc. for y and z) would be less ambiguous but would make various formulas in these notes harder to comprehend).

[3]Although it is not *hard* to show that the asserted bases are bases, a proof would require a digression that would make these notes even longer.