MAT 4930—Curves and Surfaces—Spring 2019
Non-book homework problems

1. Show that if real-valued functions f and g on $\mathbf{R}^3$ are C^∞, then so are $f+g$ and fg. (Hint for the product: induction.)

2. Let a, b be real numbers, and define $\alpha : \mathbf{R} \to \mathbf{R}^3$ by $\alpha(t) = (a\cos t, a\sin t, bt)$. Let C be the Curve parametrized by α. In class we saw that if $a > 0$, then C is a helix if $b \neq 0$, and a circle in the xy plane if $b = 0$.

(a) What is C if $a = 0$ and $b \neq 0$?

(b) What is C if $a = 0 = b$?

(c) Describe C if $a < 0$ and $b \neq 0$. For a given b, how does this C compare with the Curve we would get if a were replaced by $|a|$? Is C still a helix? If so, is its "handedness" the same as if a were replaced by $|a|$, or the opposite?

(d) Show that unless $a = 0 = b$, the curve α is a regular parametrization of C. (Since, by definition, α parametrizes C, all you need to show is that α is a regular curve.)

3. Let I be an open interval $\beta : I \to \mathbf{R}^2$ a smooth curve in the plane. Thus $\beta(t) = (\beta_1(t), \beta_2(t))$, where the $\beta_i : I \to \mathbf{R}$ are C^∞. Define $\alpha : I \to \mathbf{R}^3$ by $\alpha(t) = (t, \beta_1(t), \beta_2(t))$. The Curve $\widetilde{C}$ in $\mathbf{R}^3$ parametrized by α is called the *graph* of β. Show that α is a regular parametrization of $\widetilde{C}$, regardless of whether β is regular.

Remark. If we identify $\mathbf{R} \times \mathbf{R}^2$ with $\mathbf{R}^3$ by identifying $(x, (y, z))$ with (x, y, z), then the definition of α can be written more simply as $\alpha(t) = (t, \beta(t))$, and $\widetilde{C}$ is exactly the set of points $\{(t, \beta(t)) : t \in I\}$. The similarity of this with the definition of "graph of a real-valued function of one variable" is why $\widetilde{C}$ is called the graph of β.

Once we define "smooth Curve" in class, the upshot of the above problem will be that the *graph* $\widetilde{C}$ of a smooth curve β in $\mathbf{R}^2$ is always a smooth Curve in $\mathbf{R}^3$, whether or not the Curve C in $\mathbf{R}^2$ parametrized by β is smooth.

4. In the setup of O'Neill's problem 1.6/8 there is a 1-1 correspondence "(1)" between vector fields and 1-forms, and a 1-1 correspondence "(2)" between vector fields and 2-forms. These correspondences are also valid pointwise. (I.e. they are valid at each point $\mathbf{p} \in \mathbf{R}^3$ if you replace vector fields, 1-forms, and 2-forms, respectively, with tangent vectors at $\mathbf{p}$, cotangent vectors at $\mathbf{p}$, and what I called "2-covectors at $\mathbf{p}$" in class.)

Let $\mathbf{v_p}, \mathbf{w_p}$ be tangent vectors at $\mathbf{p} \in \mathbf{R}^3$, let $\phi_\mathbf{p}, \psi_\mathbf{p}$ be the corresponding cotangent vectors using correspondence (1), and let $\mathbf{u_p} \in T_\mathbf{p}\mathbf{R}^3$ be the tangent vector corresponding to $\phi_\mathbf{p} \wedge \psi_\mathbf{p}$ using correspondence (2). In terms of the "classical" vector operations you learned in Calculus 3 (and/or physics classes), give a simple formula for $\mathbf{u}$ in terms of

$\mathbf{v}$ and $\mathbf{w}$. (Of course, if we let the point $\mathbf{p}$ vary, the analogous formula holds for vector fields.)

5. Again using the 1-1 correspondences in O'Neill's problem 1.6/8, show the following:

(a) The fact that "$d^2 = 0$ on 0-forms", i.e. that $d(df) = 0$ for every (smooth) function $f : \mathbf{R}^3 \to \mathbf{R}$, is equivalent to the fact that $\mathrm{curl}(\mathrm{grad}\, f) = 0$ for all such f. (Here grad f is the gradient of f.)

(b) The fact that "$d^2 = 0$ on 1-forms" is equivalent to the fact that $\mathrm{div}(\mathrm{curl}(V)) = 0$ for all vector vields V.

6. (a) Let $f : \mathbf{R} \to \mathbf{R}$ be a (smooth) function, and define $\alpha : \mathbf{R} \to \mathbf{R}$ by

$$\alpha(t) = (f(t)\cos t, f(t)\sin t, f(t)). \tag{1}$$

If f is monotone, the Curve parametrized by α can reasonably be called a "conical helix". Figure out why.

For the rest of this problem, f and α are as in equation (1).

(b) Write down, in terms of f, the integral that gives the arclength function $t \mapsto s(t)$ of the curve α, based at $t = 0$.

(c) Show that α is regular provided there is no $t_0 \in \mathbf{R}$ for which $f(t_0) = f'(t_0) = 0$.

(d) For the case $f(t) = e^t$, qualitatively sketch the Curve parametrized by α. (Don't worry about drawing it to scale.)

(e) Again for the case $f(t) = e^t$, find an explicit formula for $s(t)$, solve for t in terms of s, and write down the corresponding unit-speed reparametrization β of α.

(f) What is the domain of β in part (e)? You should find that it is an interval of the form $(-b, \infty)$, where $b > 0$. What is the value of b telling you geometrically?

7. Read the following definitions, example, and remark.

Definition 1. Let $a, b \in \mathbf{R}$, with $a < b$, and let $k \geq 0$ be an integer. A continuous function $\alpha : (a, b) \to \mathbf{R}^n$ is a *piecewise regular curve* if there are finitely many points $t_0 < t_1 < \cdots < t_{N-1} < t_N$ in $[a, b]$, with $t_0 = a$ and $t_N = b$, such that

(i) α is smooth on each subinterval $(t_{j-1}, t_j), 1 \leq j \leq N$, and

(ii) for $1 \leq j \leq N$, the one-sided limit of $\alpha'|_{(t_{j-1}, t_j)}$ exists at each endpoint of (t_{j-1}, t_j) and is nonzero.

Note that every *regular* curve $\alpha : (a, b) \to \mathbf{R}^n$ is piecewise regular (just take $N = 1$ above).

More generally, if $I \subset \mathbf{R}$ is an interval (not necessarily open or bounded), we say that $\alpha : I \to \mathbf{R}^n$ is a piecewise regular curve if the restriction of α to every bounded, open subinterval of I is piecewise regular.

Note that if the interval $I \subset \mathbf{R}$ is open and $\alpha : I \to \mathbf{R}^n$ is *smooth*, it is possible for α to be piecewise regular yet not regular: at any given $t \in (a,b)$, the two one-sided limits of α' exist *and are equal*, but they may be zero.

Definition 2. A Curve $C \subset \mathbf{R}^n$ is *piecewise smooth* if it admits a piecewise regular parametrization.

Just as for curves, “smooth” is a special case of “piecewise smooth”. But typically, a piecewise-smooth Curve is a concatenation of a bunch of smooth Curve-segments (“concatenation” meaning here that the terminal point of the first segment coincides with the initial point of the second segment, etc.).

Example. Let $I \subset \mathbf{R}$ be an open interval and $\alpha : I \to \mathbf{R}^n$ a smooth curve for which $\alpha'(t)$ is zero at finitely many points $t_1 < t_2 < \dots t_k$, but is nonzero for all other t. Even though there are points at which $\alpha'(t) = \mathbf{0}$, we can reparametrize α by arclength, with the reparametrized curve being a *continuous* but not necessarily smooth map from some interval into $\mathbf{R}^n$. Specifically, let $\phi : I \to \mathbf{R}$ be the arclength function of α based at a point $t_* \in I$. Then ϕ is a strictly increasing, continuous function from I to some open interval J. It is not hard to show that the inverse of any such function is continuous. Hence the function $\beta := \alpha \circ \phi^{-1} : J \to \mathbf{R}^n$ is continuous, and a faithful reparametrization of α. However, $\phi'(t_j) = \|\alpha'(t_j)\| = 0$ for all j, so ϕ^{-1} is *not* differentiable at the points $s_j = \phi(t_j)$. The Chain Rule Theorem *does not apply* in this situation (its hypotheses are not met); the function β *may or may not* be differentiable at a given s_j.

Continuing with this example, the real-valued functions $t \mapsto \|\alpha'(t)\|$ and $s \mapsto \|\beta'(s)\|$ are smooth on $I \setminus \{t_1, \dots, t_k\}$ and $J \setminus \{s_1, \dots s_k\}$, respectively. The vector-valued functions $t \mapsto \mathbf{T}^{(\alpha)}(t) := \frac{\alpha'(t)}{\|\alpha'(t)\|}$ and $s \mapsto \mathbf{T}^{(\beta)}(s) := \beta'(s) = \mathbf{T}^{(\alpha)}(\phi^{-1}(s))$ are defined on $I \setminus \{t_1, \dots, t_k\}$ and $J \setminus \{s_1, \dots s_k\}$ respectively, and are smooth on these respective domains. The limits $\lim_{t \to t_j \pm} \mathbf{T}^{(\alpha)}(t)$ and $\lim_{s \mapsto s_j \pm} \beta'(s)$ may or may not exist, but for a given j and choice of sign in “$\pm$”, either $\lim_{t \to t_j \pm} \mathbf{T}^{(\alpha)}(t) = \lim_{s \mapsto s_j \pm} \beta'(s)$ (with both limits existing) or neither limit exists.

If all $2k$ of the one-sided limits $\lim_{s \mapsto s_j \pm} \beta'(s)$ exist (which, by the preceding, we can determine from the existence or non-existence of $\lim_{t \to t_j \pm} \mathbf{T}^{(\alpha)}(t)$), then β is piecewise regular. (No such limit, if it exists, can be zero, since if $\lim_{t \to t_j \pm} \mathbf{T}^{(\alpha)}(t)$ exists, then $\| \lim_{t \to t_j \pm} \mathbf{T}^{(\alpha)}(t)\| = \lim_{t \to t_j \pm} \|\mathbf{T}^{(\alpha)}(t)\| = \lim_{t \to t_j \pm}(1) = 1$.) In this case, the Curve C parametrized by α and β is (at least) *piecewise* smooth.

Suppose now that, additionally, α is one-to-one. Then the only one-to-one curves with image C are faithful reparametrizations of α (maps of the form $\alpha \circ h$ where h is a *continuous* bijective map from some interval J to I).[1] Recall that, by definition, a

[1]To see why “one-to-one” is relevant to the continuity of h, think about a smooth curve α that crosses

smooth Curve admits a regular parametrization γ. For such γ, both the numerator and denominator of $\frac{\gamma'(u)}{\|\gamma'(u)\|} =: \mathbf{T}^{(\gamma)}(u)$ are smooth functions, and the denominator is nowhere zero, so $\mathbf{T}^{(\gamma)}$ $\mathbf{T}^{(\gamma)}$ is continuous (in fact, smooth). Hence if our Curve C is smooth, and γ is a regular parametrization, and $u_j \in \text{domain}(\gamma)$ corresponds to $t_j \in I$, then $\lim_{t\to t_j+} \mathbf{T}^{(\alpha)}(t) = \lim_{u\to u_j+} \mathbf{T}^{(\gamma)}(u) = \lim_{u\to u_j-} \mathbf{T}^{(\gamma)}(u) = \lim_{t\to t_j-} \mathbf{T}^{(\alpha)}(t)$. Thus if α is one-to-one, *a necessary condition for C to be a smooth Curve is that* $\lim_{t\to t_j+} \mathbf{T}^{(\alpha)}(t) = \lim_{t\to t_j-} \mathbf{T}^{(\alpha)}(t)$ (equivalently, that $\lim_{t\to t_j} \mathbf{T}^{(\alpha)}(t)$ exists). Had we taken the definition of "smooth curve" (lower case) to be a map $\alpha : I \to \mathbf{R}^n$ whose *first* derivative is continuous, rather than requiring α to be C^∞ (which we did as a matter of convenience), then the necessary condition above would have been sufficient as well.

Remark. In the example above, even if the limits $\lim_{t\to t_j\pm} \frac{\alpha'(t)}{\|\alpha'(t)\|}$ exist, the hypothesis that α is C^∞ is not enough to ensure that the one-sided limits of β'' and higher-order derivatives of β exist at the points s_j. This is why, in Definition 1, we did not require that the one-sided limits of *all* derivatives of α exist at the points t_j; we would have been prevented from concluding that the reparametrization β is piecewise regular. For applications of the term "piecewise regular" in this course, it would be inconvenient to have to worry about whether the one-sided limits of higher-order derivatives exist.

Definition 3. Let $I \subset \mathbf{R}$ be an interval, let $\alpha : I \to \mathbf{R}^n$ be a piecewise regular curve, let C be the piecewise smooth Curve parametrized by α, and let $\mathbf{T}(t) = \frac{\alpha'(t)}{\|\alpha'(t)\|}$ wherever $\alpha'(t) \neq \mathbf{0}$. If $t_0 \in I$ is a point at which $\lim_{t\to t_0+} \mathbf{T}(t) \neq \lim_{t\to t_0-} \mathbf{T}(t)$, we say that α has either a *corner* or *cusp* at t_0: a cusp if $\lim_{t\to t_0+} \mathbf{T}(t) = -\lim_{t\to t_0-} \mathbf{T}(t)$, and a corner otherwise. If α is one-to-one, we correspondingly call the point $\alpha(t_0)$ a corner or cusp of C.

Note that, in the "α is one-to-one" case, if there are any corners or cusps then C is *only* piecewise smooth; C *cannot* be smooth, because of the necessary condition mentioned earlier.

8. Let $a, b \in \mathbf{R}$, with $a > 0$. In class we computed the curvature κ and torsion τ of the curve α defined by $\alpha(t) = (a\cos t, a\sin t, bt)$ (a helix if $b \neq 0$, a circle if $b = 0$). Solve for a and b in terms of κ and τ, and rewrite the formula for $\alpha(t)$ in terms of κ and τ.

9. Let $\beta : I \to \mathbf{R}^3$ be a regular curve, let λ be a positive real number, and define a curve $\gamma : I \to \mathbf{R}^3$ by $\gamma(t) = \lambda\beta(t)$.[2] The Curves parametrized by β and γ are *similar* in the sense of Euclidean geometry: one is simply a "rescaled" version of the other. (Qualitatively, the two curves have essentially the same shape, but [unless $\lambda = 1$] different size.)

(a) Let $\kappa_\beta : I \to \mathbf{R}$ and $\kappa_\gamma : I \to \mathbf{R}$ denote the curvature functions of β and

itself. You can parametrize the image by a curve $\gamma = \alpha \circ h$ that makes a sharp turn at a crossing-point instead of continuing through in the same direction, but h won't be continuous.

[2]The letter 'γ' is a lower-case gamma.

γ respectively. Find the precise relation between κ_γ and κ_β, and deduce that if κ_β is everywhere positive, then so is κ_γ.

(b) Assume that κ_β is everywhere positive, so that the torsion function $\tau_\beta : I \to \mathbf{R}$ of β is defined (and hence, by part (b), so is the torsion function $\tau_\gamma : I \to \mathbf{R}$). Find the precise relation between τ_γ and τ_β.

10. Recall that (i) a curve $\alpha : \mathbf{R} \to \mathbf{R}^n$ is *periodic* if there exists some $\rho > 0$ (called a *period* of α) such that $\alpha(t+\rho) = \alpha(t)$ for all $t \in \mathbf{R}$, and that (ii) the image of a regular periodic curve is a smooth Curve. For a periodic curve, the Chain Rule implies that $\alpha'(t+\rho) = \alpha'(t)$, and hence $\mathbf{T}(\rho) = \mathbf{T}(0)$) as well.[3] A regular periodic curve α always has a *minimal* period ρ_0, and we define the total curvature of the image Curve C is defined to be the total curvature of $\alpha|_{[0,\rho_0]}$ (cf. O'Neill problem 17(d)).

For smooth closed *plane* curves whose curvature function is strictly positive, the total curvature has topological significance. (More generally, even if the curvature is zero somewhere, it's still true that the total "plane curvature" $\int_0^\rho \tilde{\kappa}(s)\, ds$ has topological significance.) Suppose that $\alpha : \mathbf{R} \to \mathbf{R}^2$ is a regular periodic curve whose curvature function is strictly positive. Show that the total curvature of the image curve C is $2\pi n$ for some integer $n > 0$. *Hint*: Everything you need is in O'Neill problem 2.3/ 8.

Note: O'Neill problem 2.4/18 says that the total curvature of such a curve is 2π. That's correct for *simple* closed smooth curves (and is harder to show than the "$2\pi n$" above), but not for smooth closed curves that are allowed to cross themselves.

11. Let $a > 0, b \neq 0$ be real numbers, and let $\alpha : \mathbf{R} \to \mathbf{R}^3$ be the helix defined by $\alpha(t) = (a\cos t, a\sin t, bt)$.

(a) Show that a curve $\beta : \mathbf{R} \to \mathbf{R}^3$ is congruent to α if and only if there exist a point $\mathbf{p} \in \mathbf{R}^3$ and a frame $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ of $\mathbf{R}^3$ such that

$$\beta(t) = \mathbf{p} + a(\cos t\, \mathbf{e}_1 + \sin t\, \mathbf{e}_2) + bt\, \mathbf{e}_3 \ . \tag{2}$$

(b) Under what conditions on the point $\mathbf{p} \in \mathbf{R}^3$ and a frame $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ in equation (2) is there an *orientation-preserving* isometry $F : \mathbf{R}^3 \to \mathbf{R}^3$ such that $\beta = F \circ \alpha$?

(c) Let $\tilde{\mathbf{p}}$ be a point $\mathbf{R}^3$, let $\{\tilde{\mathbf{e}}_1, \tilde{\mathbf{e}}_2, \tilde{\mathbf{e}}_3\}$ be a frame of of $\mathbf{R}^3$, let $\tilde{a}, \tilde{b}$ be real numbers with $\tilde{a} > 0$, and let $\tilde{\beta} : \mathbf{R} \to \mathbf{R}^3$ be the curve defined by

$$\tilde{\beta}(t) = \tilde{\mathbf{p}} + \tilde{a}(\cos t\, \mathbf{e}_1 + \sin t\, \mathbf{e}_2) + \tilde{b}t\, \mathbf{e}_3 \quad \text{for all } t \in \mathbf{R}. \tag{3}$$

What are the most general conditions on $\tilde{\mathbf{p}}$, $\{\tilde{\mathbf{e}}_1, \tilde{\mathbf{e}}_2, \tilde{\mathbf{e}}_3\}$, $\tilde{a}$, and $\tilde{b}$, under which $\tilde{\beta}$ is congruent to α?

12. Let M be a surface in $\mathbf{R}^3$, let p be a point of M, and let $\mathcal{V}$ be an open neighborhood of p in M on which there is unit normal vector field U, and let S^M be the shape operator

[3] In class, we used the letter T for what we're now calling ρ (rho); I've changed the letter because we have an abundance of uses of 'T', and the handwritten version of "$\mathbf{T}(T)$" is hard to read.

of M determined by U. (Here, as in class, "open neighborhood of p in M" just means "open set in M that contains p".) Let $F : \mathbf{R}^3 \to \mathbf{R}^3$ be an isometry, and let $\bar{M} = F(M)$. Establish the facts below. If you get stuck on any part, assume the asserted fact in order to do later parts.

(a) The set $\bar{M}$ is a surface.

(b) For each $p \in M$, the tangent map F_{*p} carries T_pM to $T_{F(p)}\bar{M}$. (I.e. $\{F_{*p}(v) \mid v \in T_pM\} = T_{F(p)}(F(M))$.

(c) If $\mathcal{V}$ is an open subset of M, then $F(\mathcal{V})$ is an open subset of $\bar{M}$.

(d) If Z is a Euclidean vector field on an open subset $\mathcal{V} \subset M$, then the "set-theoretic vector field" F_*Z on $F(\mathcal{V})$ defined by $(F_*Z)(q) = F_*(Z(F^{-1}(q)))$ is infinitely differentiable, hence meets our criteria to be called a (true) Euclidean vector field on $F(V)$. Furthermore:

   (i) If Z is tangent to M, then F_*Z is tangent to $\bar{M}$.

   (ii) If Z is normal to M, then F_*Z is normal to $\bar{M}$.

   (ii) If $\|Z(p)\| = 1$ for all $p \in \mathcal{V}$, then $\|(F_*Z)(q)\| = 1$ for all $q \in F(\mathcal{V})$.

(e) Let U be a unit normal vector field on $\mathcal{V}$ (where "normal" means "normal to M"). By part (d), the vector field $\bar{U} := F_*U$ is a unit normal vector field on $F(\mathcal{V})$ (where "normal" means "normal to $\bar{M}$"). On $\mathcal{V}$, let S be the shape operator of M determined by U; similarly, on $F(\mathcal{V})$, let $\bar{S}$ be the shape operator of $\bar{M}$ determined by $\bar{U}$. Let $p \in \mathcal{V}$. Then:

   (i) For all $\mathbf{v} \in T_pM$, we have $F_*(S_p(\mathbf{v})) = \bar{S}_{F(p)}(F_*\mathbf{v})$.

   (ii) The principal curvatures, Gaussian curvature, and mean curvature of M' at $F(p)$ have the same values as the corresponding curvatures of M at p.

Said another way: isometries of $\mathbf{R}^3$ preserve shape operators, principal curvatures, Gaussian curvatures, and mean curvatures of surfaces. (Really we should say "preserve, up to sign," for everything except the Gaussian curvature, since the other objects are determined only up to an overall factor of ± 1.)